

On the Well-posedness of 2-D Incompressible Navier-Stokes Equations with Variable Viscosity in Critical Spaces*

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Abstract

In this paper, we first prove the local well-posedness of the 2-D incompressible Navier-Stokes equations with variable viscosity in critical Besov spaces with negative regularity indices, without smallness assumption on the variation of the density. The key is to prove for $p \in (1, 4)$ and $a \in \dot{B}_{p,1}^{\frac{2}{p}}(\mathbb{R}^2)$ that the solution mapping $\mathcal{H}_a : F \mapsto \nabla \Pi$ to the 2-D elliptic equation $\operatorname{div}((1+a)\nabla \Pi) = \operatorname{div} F$ is bounded on $\dot{B}_{p,1}^{\frac{2}{p}-1}(\mathbb{R}^2)$. More precisely, we prove that

$$\|\nabla \Pi\|_{\dot{B}_{p,1}^{\frac{2}{p}-1}} \leq C(1 + \|a\|_{\dot{B}_{p,1}^{\frac{2}{p}}})^2 \|F\|_{\dot{B}_{p,1}^{\frac{2}{p}-1}}.$$

The proof of the uniqueness of solution to (1.2) relies on a Lagrangian approach [15–17]. When the viscosity coefficient $\mu(\rho)$ is a positive constant, we prove that (1.2) is globally well-posed.

Key Words: Incompressible Navier-Stokes equations; Littlewood-Paley theory; Lagrangian coordinates; Well-posedness.

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1. Introduction

In this paper, we study the Cauchy problem of the 2-D incompressible Navier-Stokes equations with variable viscosity in critical Besov spaces

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(2\mu(\rho)\mathcal{M}(u)) + \nabla \Pi = 0, \\ \operatorname{div} u = 0, \\ (\rho, u)|_{t=0} = (\rho_0, u_0), \end{cases} \quad (1.1)$$

where ρ and $u = (u_1, u_2)$ stand for the density and velocity field, $\mathcal{M}(u) = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$, Π is a scalar pressure function, the viscosity coefficient $\mu(\rho)$ is smooth, positive on $[0, \infty)$. Throughout, we assume that the space variable x belongs to the whole space \mathbb{R}^2 .

Global weak solutions with finite energy to system (1.1) were first obtained by the Russian school [6] in the case when $\mu(\rho) = \mu > 0$ and ρ_0 is bounded away from 0. We also refer to [24] for an overview of results on weak solutions and to [18–20] for some improvements. However, the uniqueness of weak solutions is not known in general. When $\mu(\rho) = \mu > 0$ and ρ_0 is bounded away from 0, Ladyzhenskaya and Solonnikov [23] initiated the studies for unique solvability of system (1.1) in a bounded domain Ω with homogeneous Dirichlet boundary condition for u . Similar results were established by Danchin [11] in the whole space \mathbb{R}^n with initial data in the almost critical Sobolev spaces. On the other hand, from the viewpoint of physics, it is interesting to study the case for which density is discontinuous. Recently, Danchin and Mucha [17] proved by using a Lagrangian approach that the system (1.1) has a unique local solution with initial data $(\rho_0, u_0) \in L^\infty(\mathbb{R}^n) \times H^2(\mathbb{R}^n)$ if initial vacuum does not occur, see also some improvements in [21, 25].

On the other hand, if the density ρ is away from zero, we denote by $a \stackrel{\text{def}}{=} \frac{1}{\rho} - 1$ and $\tilde{\mu}(a) \stackrel{\text{def}}{=} \mu(\frac{1}{1+a})$ so that the system (1.1) can be equivalently reformulated as

$$\begin{cases} \partial_t a + u \cdot \nabla a = 0, \\ \partial_t u + u \cdot \nabla u - (1+a) \{ \operatorname{div}(2\tilde{\mu}(a)\mathcal{M}(u)) - \nabla \Pi \} = 0, \\ \operatorname{div} u = 0, \\ (a, u)|_{t=0} = (a_0, u_0). \end{cases} \quad (1.2)$$

Just as the classical Navier-Stokes equations, the system (1.2) also has a scaling. Indeed, if (a, u) solves (1.2) with initial data (a_0, u_0) , then for any $\lambda > 0$,

$$(a, u)_\lambda(t, x) \stackrel{\text{def}}{=} (a(\lambda^2 t, \lambda x), \lambda u(\lambda^2 t, \lambda x))$$

also solves (1.2) with initial data $(a_0(\lambda \cdot), \lambda u_0(\lambda \cdot))$. Moreover, the norm of $(a_0(\lambda \cdot), \lambda u_0(\lambda \cdot))$ is independent of λ in the so called critical spaces $\dot{B}_{p,1}^{\frac{2}{p}}(\mathbb{R}^2) \times \dot{B}_{p,1}^{\frac{2}{p}-1}(\mathbb{R}^2)$. In recent ten years, the French school studied the well-posedness of incompressible or compressible fluids in the framework of critical Besov spaces (see, e.g., [1, 5, 9, 10, 13]).

Motivated by [3, 4, 13–15] concerning the well-posedness of the incompressible or compressible fluids without smallness assumption on the variation of the initial density, we study the well-posedness of (1.2) in critical Besov spaces with negative regularity. More precisely, we will prove the following main theorem:

Theorem 1.1. *Let $p \in (1, 4)$, $(a_0, u_0) \in \dot{B}_{p,1}^{\frac{2}{p}}(\mathbb{R}^2) \times \dot{B}_{p,1}^{\frac{2}{p}-1}(\mathbb{R}^2)$ with $\operatorname{div} u_0 = 0$ and $1 + a_0 \geq \kappa > 0$. Assume that $\tilde{\mu}(a)$ is a smooth, positive function on $[0, \infty)$. Then (1.2) has a unique local solution $(a, u, \nabla \Pi)$ on $[0, T]$ such that*

$$\begin{aligned} a &\in C([0, T]; \dot{B}_{p,1}^{\frac{2}{p}}(\mathbb{R}^2)) \cap \tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{2}{p}}(\mathbb{R}^2)), \quad \nabla \Pi \in L_T^1(\dot{B}_{p,1}^{\frac{2}{p}-1}(\mathbb{R}^2)), \\ u &\in C([0, T]; \dot{B}_{p,1}^{\frac{2}{p}-1}(\mathbb{R}^2)) \cap \tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{2}{p}-1}(\mathbb{R}^2)) \cap L_T^1(\dot{B}_{p,1}^{\frac{2}{p}+1}(\mathbb{R}^2)). \end{aligned} \quad (1.3)$$

Moreover, if $\tilde{\mu}(a)$ is a positive constant and $a_0 \in \dot{B}_{p,1}^{\frac{2}{p}}(\mathbb{R}^2)$, then (1.2) has a unique global solution $(a, u, \nabla \Pi)$ such that for any $t > 0$:

$$\|a\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{2}{p}})} + \|u\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{2}{p}-1})} + \|u\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}+1})} + \|\nabla \Pi\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} \leq C \exp \left\{ C \exp \left(C t^{\frac{1}{2}} \right) \right\}. \quad (1.4)$$

for some time independent constant C .

As in [3, 4, 12, 14], a central problem to prove the local well-posedness part of Theorem 1.1 is the estimate of the pressure function. However, in the particular case when the spatial dimension is two, we could prove for $p \in (1, 4)$ and $a \in \dot{B}_{p,1}^{\frac{2}{p}}(\mathbb{R}^2)$ that the solution mapping $\mathcal{H}_a : F \mapsto \nabla \Pi$ to the 2-D elliptic equation

$$\operatorname{div}((1 + a)\nabla \Pi) = \operatorname{div} F \quad \text{in } \mathbb{R}^2 \quad (1.5)$$

is bounded on $\dot{B}_{p,1}^{\frac{2}{p}-1}(\mathbb{R}^2)$ (see Proposition 3.1 below). This in some sense explains that the 2-D problem is a critical problem. We shall also mention that the proof of the uniqueness of solution to (1.2) relies on a Lagrangian approach [15–17]. Finally, if the viscosity coefficient $\mu(\rho) = \tilde{\mu}(a) \equiv 1$, then the systems (1.1) and (1.2) turn respectively into

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0, \\ \rho(\partial_t u + u \cdot \nabla u) - \Delta u + \nabla \Pi = 0, \\ \operatorname{div} u = 0, \\ (\rho, u)|_{t=0} = (\rho_0, u_0), \end{cases} \quad (1.6)$$

and

$$\begin{cases} \partial_t a + u \cdot \nabla a = 0, \\ \partial_t u + u \cdot \nabla u - (1+a)\Delta u + (1+a)\nabla \Pi = 0, \\ \operatorname{div} u = 0, \\ (a, u)|_{t=0} = (a_0, u_0). \end{cases} \quad (1.7)$$

Notice that $u_0 \in \dot{B}_{p,1}^{\frac{2}{p}-1}(\mathbb{R}^2)$ is not of finite energy for $p \in (2, 4)$. For this, we set $\bar{u} \stackrel{\text{def}}{=} u - u_F$ with $u_F(t) \stackrel{\text{def}}{=} e^{(t-t_1)\Delta} u(t_1)$. Then we deduce from (1.6) that $(\rho, \bar{u}, \nabla \Pi)$ solves

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho(u_F + \bar{u})) = 0, \\ \rho(\partial_t \bar{u} + (u_F + \bar{u}) \cdot \nabla \bar{u}) - \Delta \bar{u} + \nabla \Pi = G, \\ \operatorname{div} \bar{u} = 0, \\ \rho|_{t=t_1} = \rho(t_1), \quad \bar{u}|_{t=t_1} = 0 \end{cases} \quad (1.8)$$

with $G \stackrel{\text{def}}{=} (1-\rho)\Delta u_F - \rho u_F \cdot \nabla u_F - \rho \bar{u} \cdot \nabla u_F$. By using the method in [3, 4], we could present the L^2 energy estimate for \bar{u} to prove the global well-posedness part of Theorem 1.1.

The remainder of this paper is organized as follows. In Section 2, we present some basic facts on Littlewood–Paley analysis and introduce several technical lemmas, then we present the estimates to the free transport equation. In Section 3, we study some linear elliptic and parabolic equations with rough coefficients in the framework of critical Besov spaces. In Section 4, we complete the proof of the local well-posedness part of Theorem 1.1. In Section 5, we present the energy estimate for \bar{u} in the L^2 framework to prove the global well-posedness of (1.2) with $\mu(\rho) \equiv 1$.

Notations: For two operators A and B , we denote $[A, B] = AB - BA$ the commutator between A and B . The letter C stands for a generic constant whose meaning is clear from the context. We sometimes write $a \lesssim b$ instead of $a \leq Cb$. For $p \in [1, \infty]$, the conjugate index p' is determined by $\frac{1}{p} + \frac{1}{p'} = 1$. The Fourier transform of u is denoted either by \hat{u} or $\mathcal{F}u$, the inverse by $\mathcal{F}^{-1}u$. The notation \mathcal{P} stands for the Leray projector on the divergence free vector fields, while $\mathcal{Q} = \text{Id} - \mathcal{P}$ stands for the projector on the gradient type vector fields.

For X a Banach space and I an interval of \mathbb{R} , we denote by $C(I; X)$ the set of continuous functions on I with values in X . For $q \in [1, +\infty]$, $L^q(I; X)$ stands for the set of measurable functions on I with values in X , such that $t \mapsto \|f(t)\|_X$ belongs to $L^q(I)$. For short, we sometimes write $L_T^q(X)$ instead of $L^q((0, T); X)$.

2. Preliminaries

We first recall some basic facts on Littlewood-Paley theory (see [7] for instance). Let χ, φ be two smooth radial functions valued in the interval $[0, 1]$, the support of χ be the ball $\mathcal{B} = \{\xi \in \mathbb{R}^2 : |\xi| \leq \frac{4}{3}\}$ while the support of φ be the annulus $\mathcal{C} = \{\xi \in \mathbb{R}^2 : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$, and satisfy

$$\begin{aligned} \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) &= 1 \quad \text{for } \xi \in \mathbb{R}^2 \setminus \{0\}; \\ \chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) &= 1 \quad \text{for } \xi \in \mathbb{R}^2. \end{aligned}$$

Denote by $h \stackrel{\text{def}}{=} \mathcal{F}^{-1}\varphi$ and $\tilde{h} \stackrel{\text{def}}{=} \mathcal{F}^{-1}\chi$, the homogeneous dyadic blocks $\dot{\Delta}_j$ and the homogeneous low-frequency cutoff operators \dot{S}_j are defined for all $j \in \mathbb{Z}$ by

$$\begin{aligned} \dot{\Delta}_j u &= \varphi(2^{-j}D)u = 2^{2j} \int_{\mathbb{R}^2} h(2^j y) u(x - y) dy, \\ \dot{S}_j u &= \chi(2^{-j}D)u = 2^{2j} \int_{\mathbb{R}^2} \tilde{h}(2^j y) u(x - y) dy. \end{aligned}$$

Denote by $\mathcal{S}'_h(\mathbb{R}^2)$ the space of tempered distributions u such that

$$\lim_{j \rightarrow -\infty} \dot{S}_j u = 0 \quad \text{in } \mathcal{S}'(\mathbb{R}^2).$$

Then we have the formal decomposition

$$u = \sum_{j \in \mathbb{Z}} \dot{\Delta}_j u, \quad \forall u \in \mathcal{S}'_h(\mathbb{R}^2).$$

Moreover, the Littlewood-Paley decomposition satisfies the property of almost orthogonality:

$$\dot{\Delta}_k \dot{\Delta}_j u \equiv 0 \text{ if } |k - j| \geq 2, \text{ and } \dot{\Delta}_k (\dot{S}_{j-1} u \dot{\Delta}_j v) \equiv 0 \text{ if } |k - j| \geq 5.$$

Now we recall the definition of homogeneous Besov spaces from [7].

Definition 2.1. Let $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$. The homogeneous Besov space $\dot{B}_{p,r}^s(\mathbb{R}^2)$ consists of all the distributions u in $\mathcal{S}'(\mathbb{R}^2)$ such that

$$\|u\|_{\dot{B}_{p,r}^s} \stackrel{\text{def}}{=} \left\| \left(2^{js} \|\dot{\Delta}_j u\|_{L^p} \right)_{j \in \mathbb{Z}} \right\|_{l^r} < \infty.$$

Remark 2.1. With some slight modifications, we can also define inhomogeneous Besov spaces. Indeed, for $u \in \mathcal{S}'(\mathbb{R}^2)$, we define

$$\begin{aligned} \Delta_j u &= \dot{\Delta}_j u, \quad \forall j \geq 0; \quad \Delta_{-1} u = \dot{S}_0 u; \quad \Delta_j u = 0, \quad \forall j \leq -2; \\ \text{and } S_j u &= \sum_{j' \leq j-1} \Delta_{j'} u. \end{aligned}$$

Then the inhomogeneous Besov space $B_{p,r}^s(\mathbb{R}^2)$ consists of all the distributions u in $\mathcal{S}'(\mathbb{R}^2)$ such that

$$\|u\|_{B_{p,r}^s} \stackrel{\text{def}}{=} \left\| \left(2^{js} \|\Delta_j u\|_{L^p} \right)_{j \geq -1} \right\|_{l^r} < \infty.$$

Remark 2.2. Let $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$. Then there exists a positive constant C such that u belongs to $\dot{B}_{p,r}^s(\mathbb{R}^2)$ if and only if there exists $\{c_{j,r}\}_{j \in \mathbb{Z}}$ such that $c_{j,r} \geq 0$, $\|c_{j,r}\|_{l^r} = 1$ and

$$\|\dot{\Delta}_j u\|_{L^p} \leq C c_{j,r} 2^{-js} \|u\|_{\dot{B}_{p,r}^s}, \quad \forall j \in \mathbb{Z}.$$

For simplicity, we denote $d_j \stackrel{\text{def}}{=} c_{j,1}$ and $c_j \stackrel{\text{def}}{=} c_{j,2}$.

To gain a better description of the regularization effect to the transport-diffusion equation, we should use the Chemin-Lerner type norms(see [7]):

Definition 2.2. Let $s \in \mathbb{R}$ and $0 < T \leq +\infty$. We define

$$\|u\|_{\tilde{L}_T^\sigma(\dot{B}_{p,r}^s)} \stackrel{\text{def}}{=} \left(\sum_{j \in \mathbb{Z}} 2^{jrs} \left(\int_0^T \|\dot{\Delta}_j u(t)\|_{L^p}^\sigma dt \right)^{\frac{r}{\sigma}} \right)^{\frac{1}{r}}$$

for $p \in [1, \infty]$, $r, \sigma \in [1, \infty)$, and with the standard modification for $r = \infty$ or $\sigma = \infty$.

The following lemmas will be repeatedly used in this paper (see [7]).

Lemma 2.1. *Let $\mathcal{C} \subset \mathbb{R}^2$ be an annulus and $\mathcal{B} \subset \mathbb{R}^2$ be a ball. There exists a positive constant C such that for any $0 \leq k \in \mathbb{Z}$, any $\lambda > 0$, any smooth homogeneous function σ of degree m , any $1 \leq p \leq q \leq \infty$, and any function $u \in L^p$, we have*

$$\text{supp } \hat{u} \subset \lambda \mathcal{B} \implies \|D^k u\|_{L^q} \stackrel{\text{def}}{=} \sum_{|\alpha|=k} \|\partial^\alpha u\|_{L^q} \leq C^{k+1} \lambda^{k+2(\frac{1}{p}-\frac{1}{q})} \|u\|_{L^p},$$

$$\text{supp } \hat{u} \subset \lambda \mathcal{C} \implies C^{-k-1} \lambda^k \|u\|_{L^p} \leq \|D^k u\|_{L^p} \leq C^{k+1} \lambda^k \|u\|_{L^p},$$

$$\text{supp } \hat{u} \subset \lambda \mathcal{C} \implies \|\sigma(D)u\|_{L^q} \leq C_{\sigma,m} \lambda^{m+2(\frac{1}{p}-\frac{1}{q})} \|u\|_{L^p}.$$

Lemma 2.2. *Let $\mathcal{C} \subset \mathbb{R}^2$ be an annulus. Then there exist positive constants c and C , such that for any $1 \leq p \leq \infty$ and $\lambda > 0$, we have*

$$\text{supp } \hat{u} \subset \lambda \mathcal{C} \implies \|e^{t\Delta} u\|_{L^p} \leq C e^{-ct\lambda^2} \|u\|_{L^p}.$$

On the other hand, it has been demonstrated that the Bony's decomposition [7, 8] is very effective to deal with nonlinear problems. Here, we recall the Bony's decomposition in the homogeneous context:

$$uv = \dot{T}_u v + \dot{T}_v u + \dot{R}(u, v) = \dot{T}_u v + \dot{T}'_v u,$$

where

$$\begin{aligned} \dot{T}_u v &\stackrel{\text{def}}{=} \sum_{j \in \mathbb{Z}} \dot{S}_{j-1} u \dot{\Delta}_j v, \quad \dot{R}(u, v) \stackrel{\text{def}}{=} \sum_{j \in \mathbb{Z}} \dot{\Delta}_j u \tilde{\dot{\Delta}}_j v \quad \text{with} \quad \tilde{\dot{\Delta}}_j v \stackrel{\text{def}}{=} \sum_{|j'-j| \leq 1} \dot{\Delta}_{j'} v, \\ \text{and} \quad \dot{T}'_v u &\stackrel{\text{def}}{=} \dot{T}_v u + \dot{R}(u, v) = \sum_{j \in \mathbb{Z}} \dot{\Delta}_j u \dot{S}_{j+2} v. \end{aligned}$$

In the sequel, we should frequently use the following product laws [26].

Lemma 2.3. *Let $1 \leq p, q \leq \infty$, $s_1 \leq \frac{2}{q}$, $s_2 \leq 2 \min\{\frac{1}{p}, \frac{1}{q}\}$ and $s_1 + s_2 > 2 \max\{0, \frac{1}{p} + \frac{1}{q} - 1\}$.*

Then there holds

$$\|ab\|_{\dot{B}_{p,1}^{s_1+s_2-\frac{2}{q}}} \lesssim \|a\|_{\dot{B}_{q,1}^{s_1}} \|b\|_{\dot{B}_{p,1}^{s_2}}, \quad \forall (a, b) \in \dot{B}_{q,1}^{s_1}(\mathbb{R}^2) \times \dot{B}_{p,1}^{s_2}(\mathbb{R}^2).$$

Remark 2.3. We shall frequently use the fact that $\dot{B}_{p,1}^{\frac{2}{p}}(\mathbb{R}^2)$ is an algebra for $p \in [1, \infty)$ and that $\|ab\|_{\dot{B}_{p,1}^{\frac{2}{p}-1}} \lesssim \|a\|_{\dot{B}_{p,1}^{\frac{2}{p}}} \|b\|_{\dot{B}_{p,1}^{\frac{2}{p}-1}}$ for $p \in [1, 4)$.

Let us also recall the following commutator estimates (see [7, Lemma 2.100] for instance).

Lemma 2.4. *Let $(p, q) \in [1, \infty]^2$, $-1 - 2 \min\{\frac{1}{p}, \frac{1}{q'}\} < s \leq 1 + 2 \min\{\frac{1}{p}, \frac{1}{q}\}$, $a \in \dot{B}_{q,1}^s(\mathbb{R}^2)$ and $u \in \dot{B}_{p,1}^{\frac{2}{p}+1}(\mathbb{R}^2)$ with $\operatorname{div} u = 0$. Then there holds*

$$\|[u \cdot \nabla, \dot{\Delta}_j]a\|_{L^q} \lesssim d_j 2^{-js} \|u\|_{\dot{B}_{p,1}^{\frac{2}{p}+1}} \|a\|_{\dot{B}_{q,1}^s}.$$

Motivated by [3, 4, 13], we need the following proposition to deal with the transport equation in (1.2).

Proposition 2.1. *Let $(p, q) \in [1, \infty]^2$, $\frac{1}{q} - \frac{1}{p} \leq \frac{1}{2}$, $a_0 \in \dot{B}_{q,1}^{\frac{2}{q}}(\mathbb{R}^2)$ and $u \in L_T^1(\dot{B}_{p,1}^{\frac{2}{p}+1}(\mathbb{R}^2))$ with $\operatorname{div} u = 0$. If $a \in C([0, T]; \dot{B}_{q,1}^{\frac{2}{q}}(\mathbb{R}^2))$ solves*

$$\partial_t a + u \cdot \nabla a = 0, \quad (t, x) \in (0, T] \times \mathbb{R}^2$$

with initial data a_0 , then there holds for $t \in [0, T]$

$$\|a\|_{\tilde{L}_t^\infty(\dot{B}_{q,1}^{\frac{2}{q}})} \leq \|a_0\|_{\dot{B}_{q,1}^{\frac{2}{q}}} e^{CU(t)}, \quad (2.1)$$

and

$$\|a - \dot{S}_m a\|_{\tilde{L}_t^\infty(\dot{B}_{q,1}^{\frac{2}{q}})} \leq \sum_{j \geq m} 2^{\frac{2j}{q}} \|\dot{\Delta}_j a_0\|_{L^q} + \|a_0\|_{\dot{B}_{q,1}^{\frac{2}{q}}} (e^{CU(t)} - 1), \quad (2.2)$$

with $U(t) \stackrel{\text{def}}{=} \|u\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}+1})}$.

Proof. We first get by applying $\dot{\Delta}_j$ to the transport equation

$$(\partial_t + u \cdot \nabla) \dot{\Delta}_j a = [u \cdot \nabla, \dot{\Delta}_j] a.$$

Since $\nabla u \in L_T^1(L^\infty)$ with $\operatorname{div} u = 0$, we get by using classical L^q estimate for transport equation that

$$\|\dot{\Delta}_j a\|_{L_t^\infty(L^q)} \leq \|\dot{\Delta}_j a_0\|_{L^q} + \int_0^t \|[u \cdot \nabla, \dot{\Delta}_j] a\|_{L^q} d\tau,$$

from which and Lemma 2.4, we get for $\frac{1}{q} - \frac{1}{p} \leq \frac{1}{2}$

$$2^{\frac{2j}{q}} \|\dot{\Delta}_j a\|_{L_t^\infty(L^q)} \leq 2^{\frac{2j}{q}} \|\dot{\Delta}_j a_0\|_{L^q} + C \int_0^t d_j(\tau) \|u(\tau)\|_{\dot{B}_{p,1}^{\frac{2}{p}+1}} \|a(\tau)\|_{\dot{B}_{q,1}^{\frac{2}{q}}} d\tau. \quad (2.3)$$

Taking summation for $j \in \mathbb{Z}$ and then using Gronwall's inequality implies (2.1). Then substituting (2.1) into (2.3) results in

$$2^{\frac{2j}{q}} \|\dot{\Delta}_j a\|_{L_t^\infty(L^q)} \leq 2^{\frac{2j}{q}} \|\dot{\Delta}_j a_0\|_{L^q} + C \|a_0\|_{\dot{B}_{q,1}^{\frac{2}{q}}} \int_0^t d_j(\tau) U'(\tau) e^{CU(\tau)} d\tau.$$

Summing up the above inequality on $\{j \geq m\}$ leads to (2.2). \square

Remark 2.4. Let $p, q, U(t)$ be determined by Proposition 2.1. Then in the framework of inhomogeneous Besov spaces, there similarly holds

$$\begin{aligned} \|a\|_{\tilde{L}_t^\infty(B_{q,1}^{\frac{2}{q}})} &\leq \|a_0\|_{B_{q,1}^{\frac{2}{q}}} e^{CU(t)}, \\ \|a - S_m a\|_{\tilde{L}_t^\infty(B_{q,1}^{\frac{2}{q}})} &\leq \sum_{j \geq m} 2^{\frac{2j}{q}} \|\dot{\Delta}_j a_0\|_{L^q} + \|a_0\|_{B_{q,1}^{\frac{2}{q}}} (e^{CU(t)} - 1). \end{aligned}$$

3. Linear system with rough coefficients

In this section, we study some linear elliptic and parabolic equations with rough coefficients in the L^p framework. We need the following commutator estimates of integral type:

Lemma 3.1. (i) Let $(p, q) \in (\frac{1+\sqrt{17}}{4}, 2] \times [1, \infty)$ with $\frac{1}{p} - \frac{1}{q} \leq \frac{1}{2}$. Then we have

$$I_j \stackrel{\text{def}}{=} \int_{\mathbb{R}^2} \operatorname{div} ([\dot{\Delta}_j, a] \nabla \Pi) \cdot |\dot{\Delta}_j \Pi|^{p-2} \dot{\Delta}_j \Pi dx \lesssim d_j 2^{j(2-\frac{2}{p})} \|a\|_{\dot{B}_{q,1}^{\frac{2}{q}}} \|\nabla \Pi\|_{L^2} \|\dot{\Delta}_j \Pi\|_{L^p}^{p-1}. \quad (3.1)$$

(ii) For $p \in (1, 4)$, we alternatively have

$$I_j \lesssim d_j 2^{j(2-\frac{2}{p})} \|a\|_{\dot{B}_{p,1}^{\frac{2}{p}}} \|\nabla \Pi\|_{L^2} \|\dot{\Delta}_j \Pi\|_{L^p}^{p-1}, \quad p \in (1, 2), \quad (3.2)$$

$$I_j \lesssim d_j 2^{j(2-\frac{2}{p})} \|a\|_{\dot{B}_{p,1}^{\frac{2}{p}}} \|\nabla \Pi\|_{\dot{B}_{p,2}^{\frac{2}{p}-1}} \|\dot{\Delta}_j \Pi\|_{L^p}^{p-1}, \quad p \in [2, 4). \quad (3.3)$$

Proof. (i) Note that we could not directly use integration by parts. For this, we first get by using Bony's decomposition

$$\begin{aligned} I_j &= \int_{\mathbb{R}^2} \operatorname{div} ([\dot{\Delta}_j, \dot{T}_a] \nabla \Pi) \cdot |\dot{\Delta}_j \Pi|^{p-2} \dot{\Delta}_j \Pi dx + \int_{\mathbb{R}^2} \operatorname{div} \dot{\Delta}_j (\dot{T}'_{\nabla \Pi} a) \cdot |\dot{\Delta}_j \Pi|^{p-2} \dot{\Delta}_j \Pi dx \\ &\quad - \int_{\mathbb{R}^2} \operatorname{div} (\dot{T}'_{\dot{\Delta}_j \nabla \Pi} a) \cdot |\dot{\Delta}_j \Pi|^{p-2} \dot{\Delta}_j \Pi dx \stackrel{\text{def}}{=} I_j^1 + I_j^2 + I_j^3. \end{aligned}$$

By the definition of Bony's decomposition, we have

$$[\dot{\Delta}_j, \dot{T}_a] \nabla \Pi = -2^{2j} \sum_{|j'-j| \leq 4} \int_{\mathbb{R}^2} h(2^j y) \dot{\Delta}_{j'} \nabla \Pi(x-y) dy \int_0^1 y \cdot \nabla \dot{S}_{j'-1} a(x-\tau y) d\tau, \quad (3.4)$$

from which, we get by using Hölder inequality and Lemma 2.1 that

$$\|[\dot{\Delta}_j, \dot{T}_a]\nabla\Pi\|_{L^p} \lesssim 2^{-j} \sum_{|j'-j|\leq 4} \|\nabla \dot{S}_{j'-1}a\|_{L^{\frac{2p}{2-p}}} \|\dot{\Delta}_{j'}\nabla\Pi\|_{L^2} \lesssim d_j 2^{j(1-\frac{2}{p})} \|a\|_{\dot{B}_{q,1}^{\frac{2}{q}}} \|\nabla\Pi\|_{L^2},$$

where we used $\|\nabla \dot{S}_{j'-1}a\|_{L^{\frac{2p}{2-p}}} \lesssim d_{j'} 2^{j'(2-\frac{2}{p})} \|a\|_{\dot{B}_{q,1}^{\frac{2}{q}}}$ for $p > 1$ and $\frac{1}{p} - \frac{1}{q} \leq \frac{1}{2}$. Note that $[\dot{\Delta}_j, \dot{T}_a]\nabla\Pi$ is spectrally supported in an annulus of size 2^j . Whence we infer

$$I_j^1 \lesssim d_j 2^{j(2-\frac{2}{p})} \|a\|_{\dot{B}_{q,1}^{\frac{2}{q}}} \|\nabla\Pi\|_{L^2} \|\dot{\Delta}_j\Pi\|_{L^p}^{p-1}. \quad (3.5)$$

Owing to the localization properties of the Littlewood-Paley decomposition, we have

$$\dot{\Delta}_j(\dot{T}'_{\nabla\Pi}a) = \sum_{j'\geq j-3} \dot{\Delta}_j(\dot{\Delta}_{j'}a\dot{S}_{j'+2}\nabla\Pi).$$

If $q \geq 2$, we denote $\frac{1}{\gamma} \stackrel{\text{def}}{=} \frac{1}{2} + \frac{1}{q} \geq \frac{1}{p}$ and apply Lemma 2.1 to obtain

$$\|\dot{\Delta}_j(\dot{T}'_{\nabla\Pi}a)\|_{L^p} \lesssim 2^{2j(\frac{1}{\gamma}-\frac{1}{p})} \sum_{j'\geq j-3} \|\dot{\Delta}_{j'}a\|_{L^q} \|\dot{S}_{j'+2}\nabla\Pi\|_{L^2} \lesssim d_j 2^{j(1-\frac{2}{p})} \|a\|_{\dot{B}_{q,1}^{\frac{2}{q}}} \|\nabla\Pi\|_{L^2}.$$

While if $q < 2$, the embedding $\dot{B}_{q,1}^{\frac{2}{q}}(\mathbb{R}^2) \hookrightarrow \dot{B}_{2,1}^1(\mathbb{R}^2)$ ensures that the above inequality still holds. Thus we obtain

$$I_j^2 \lesssim d_j 2^{j(2-\frac{2}{p})} \|a\|_{\dot{B}_{q,1}^{\frac{2}{q}}} \|\nabla\Pi\|_{L^2} \|\dot{\Delta}_j\Pi\|_{L^p}^{p-1}. \quad (3.6)$$

For I_j^3 , we write

$$\dot{T}'_{\dot{\Delta}_j\nabla\Pi}a = \sum_{j'-j=-1,-2} \dot{\Delta}_{j'}a\dot{S}_{j'+2}\dot{\Delta}_j\nabla\Pi + \sum_{j'\geq j} \dot{\Delta}_{j'}a\dot{\Delta}_j\nabla\Pi,$$

from which, we get by applying Lemma 8 in Appendix B of [14]

$$\begin{aligned} I_j^3 &= - \sum_{j'-j=-1,-2} \int_{\mathbb{R}^2} \operatorname{div} (\dot{\Delta}_{j'}a\dot{S}_{j'+2}\dot{\Delta}_j\nabla\Pi) \cdot |\dot{\Delta}_j\Pi|^{p-2} \dot{\Delta}_j\Pi dx \\ &\quad + (p-1) \sum_{j'\geq j} \int_{\mathbb{R}^2} \dot{\Delta}_{j'}a |\dot{\Delta}_j\nabla\Pi|^2 \cdot |\dot{\Delta}_j\Pi|^{p-2} dx \stackrel{\text{def}}{=} I_j^{3,1} + I_j^{3,2}. \end{aligned}$$

Then it is easy to observe for $\frac{1}{p} - \frac{1}{q} \leq \frac{1}{2}$ that

$$\begin{aligned} I_j^{3,1} &\lesssim 2^j \sum_{j'-j=-1,-2} \|\dot{\Delta}_{j'}a\|_{L^{\frac{2p}{2-p}}} \|\dot{\Delta}_j\nabla\Pi\|_{L^2} \|\dot{\Delta}_j\Pi\|_{L^p}^{p-1} \\ &\lesssim d_j 2^{j(2-\frac{2}{p})} \|a\|_{\dot{B}_{q,1}^{\frac{2}{q}}} \|\nabla\Pi\|_{L^2} \|\dot{\Delta}_j\Pi\|_{L^p}^{p-1}. \end{aligned} \quad (3.7)$$

While the assumption $p \in (\frac{1+\sqrt{17}}{4}, 2]$ ensures that $\frac{1}{p-1} < \frac{2p}{2-p}$. In the case when $\max\{p, \frac{1}{p-1}\} < q \leq \frac{2p}{2-p}$, we have $(p-2)q' + 1 > 0$ so that we could use a similar approximate argument as the proof of Lemma A.5 in the appendix of [9] to obtain

$$\begin{aligned} & \left\| |\dot{\Delta}_j \nabla \Pi|^2 \cdot |\dot{\Delta}_j \Pi|^{p-2} \right\|_{L^{q'}}^{q'} \\ &= \int_{\mathbb{R}^2} |\dot{\Delta}_j \Pi|^{(p-2)q'} \dot{\Delta}_j \nabla \Pi \cdot \dot{\Delta}_j \nabla \Pi |\dot{\Delta}_j \nabla \Pi|^{2q'-2} dx \\ &= -\frac{1}{(p-2)q' + 1} \int_{\mathbb{R}^2} |\dot{\Delta}_j \Pi|^{(p-2)q'} \dot{\Delta}_j \Pi \cdot \operatorname{div} (|\dot{\Delta}_j \nabla \Pi|^{2q'-2}) dx. \end{aligned}$$

Denoting $\frac{1}{r} \stackrel{\text{def}}{=} \frac{1}{p} - \frac{1}{q} \leq \frac{1}{2}$ and using Hölder inequality and Lemma 2.1 gives

$$\begin{aligned} & \left\| |\dot{\Delta}_j \nabla \Pi|^2 \cdot |\dot{\Delta}_j \Pi|^{p-2} \right\|_{L^{q'}}^{q'} \\ & \lesssim \left\| |\dot{\Delta}_j \Pi|^{(p-2)q'+1} \right\|_{L^{\frac{p}{(p-2)q'+1}}} \left\| |\dot{\Delta}_j \nabla \Pi|^{q'-1} \right\|_{L^{\frac{p}{q'-1}}} \left\| |\dot{\Delta}_j \nabla \Pi|^{q'-1} \right\|_{L^{\frac{r}{q'-1}}} \left\| \nabla^2 \dot{\Delta}_j \Pi \right\|_{L^r} \\ & \lesssim 2^{jq'(2-\frac{2}{p}+\frac{2}{q})} \left\| \dot{\Delta}_j \Pi \right\|_{L^p}^{(p-1)q'} \left\| \nabla \dot{\Delta}_j \Pi \right\|_{L^2}^{q'}, \end{aligned}$$

which implies

$$\begin{aligned} I_j^{3,2} & \lesssim \sum_{j' \geq j} \left\| \dot{\Delta}_{j'} a \right\|_{L^q} \left\| |\dot{\Delta}_j \nabla \Pi|^2 \cdot |\dot{\Delta}_j \Pi|^{p-2} \right\|_{L^{q'}}^{q'} \\ & \lesssim d_j 2^{j(2-\frac{2}{p})} \|a\|_{\dot{B}_{q,1}^{\frac{2}{q}}} \left\| \nabla \Pi \right\|_{L^2} \left\| \dot{\Delta}_j \Pi \right\|_{L^p}^{p-1}. \end{aligned} \quad (3.8)$$

Similarly, (3.8) is valid for $q \leq \max\{p, \frac{1}{p-1}\}$ according to embedding. Summing up the inequalities (3.5)–(3.8) results in (3.1).

(ii) We first consider the case when $p \in (1, 2)$. From (3.5) and (3.6), we infer

$$I_j^1 + I_j^2 \lesssim d_j 2^{j(2-\frac{2}{p})} \|a\|_{\dot{B}_{p,1}^{\frac{2}{p}}} \left\| \nabla \Pi \right\|_{L^2} \left\| \dot{\Delta}_j \Pi \right\|_{L^p}^{p-1}. \quad (3.9)$$

For I_j^3 , we alternatively get

$$\begin{aligned} I_j^3 & \lesssim \sum_{j' \geq j-2} 2^{j'} \left\| \dot{\Delta}_{j'} a \right\|_{L^p} \left\| \dot{\Delta}_j \nabla \Pi \right\|_{L^\infty} \left\| \dot{\Delta}_j \Pi \right\|_{L^p}^{p-1} \\ & \lesssim d_j 2^{j(2-\frac{2}{p})} \|a\|_{\dot{B}_{p,1}^{\frac{2}{p}}} \left\| \nabla \Pi \right\|_{L^2} \left\| \dot{\Delta}_j \Pi \right\|_{L^p}^{p-1}, \end{aligned}$$

which together with (3.9) implies (3.2).

Next, if $p \in [2, 4)$, using integration by parts leads to

$$I_j = -(p-1) \int_{\mathbb{R}^2} [\dot{\Delta}_j, a] \nabla \Pi \cdot |\dot{\Delta}_j \Pi|^{p-2} \nabla \dot{\Delta}_j \Pi dx \lesssim 2^j \|[\dot{\Delta}_j, a] \nabla \Pi\|_{L^p} \left\| \dot{\Delta}_j \Pi \right\|_{L^p}^{p-1}. \quad (3.10)$$

Applying again Bony's decomposition gives

$$[\dot{\Delta}_j, a] \nabla \Pi = [\dot{\Delta}_j, \dot{T}_a] \nabla \Pi + \dot{\Delta}_j (\dot{T}_{\nabla \Pi} a) + \dot{\Delta}_j (\dot{R}(a, \nabla \Pi)) - \dot{T}'_{\dot{\Delta}_j \nabla \Pi} a.$$

Thanks to (3.4), we get by using Young's inequality and Lemma 2.1 that

$$\|[\dot{\Delta}_j, \dot{T}_a] \nabla \Pi\|_{L^p} \lesssim 2^{-j} \sum_{|j'-j| \leq 4} \|\nabla \dot{S}_{j'-1} a\|_{L^\infty} \|\dot{\Delta}_{j'} \nabla \Pi\|_{L^p} \lesssim d_j 2^{j(1-\frac{2}{p})} \|a\|_{\dot{B}_{p,1}^{\frac{2}{p}}} \|\nabla \Pi\|_{\dot{B}_{p,2}^{\frac{2}{p}-1}}.$$

Using again Lemma 2.1, we arrive at

$$\begin{aligned} \|\dot{\Delta}_j (\dot{T}_{\nabla \Pi} a)\|_{L^p} &\lesssim \sum_{|j'-j| \leq 4} \|\dot{\Delta}_{j'} a\|_{L^p} \|\dot{S}_{j'-1} \nabla \Pi\|_{L^\infty} \lesssim d_j 2^{j(1-\frac{2}{p})} \|a\|_{\dot{B}_{p,1}^{\frac{2}{p}}} \|\nabla \Pi\|_{\dot{B}_{p,2}^{\frac{2}{p}-1}}, \\ \|\dot{T}'_{\dot{\Delta}_j \nabla \Pi} a\|_{L^p} &\lesssim \sum_{j' \geq j-2} \|\dot{\Delta}_{j'} a\|_{L^p} \|\dot{\Delta}_j \nabla \Pi\|_{L^\infty} \lesssim d_j 2^{j(1-\frac{2}{p})} \|a\|_{\dot{B}_{p,1}^{\frac{2}{p}}} \|\nabla \Pi\|_{\dot{B}_{p,2}^{\frac{2}{p}-1}}, \end{aligned}$$

and for $p \in [2, 4)$

$$\|\dot{\Delta}_j (\dot{R}(a, \nabla \Pi))\|_{L^p} \lesssim 2^{\frac{2j}{p}} \sum_{j' \geq j-3} \|\dot{\Delta}_{j'} a\|_{L^p} \|\widetilde{\dot{\Delta}_{j'}} \nabla \Pi\|_{L^p} \lesssim d_j 2^{j(1-\frac{2}{p})} \|a\|_{\dot{B}_{p,1}^{\frac{2}{p}}} \|\nabla \Pi\|_{\dot{B}_{p,2}^{\frac{2}{p}-1}}.$$

Thus we obtain

$$\|[\dot{\Delta}_j, a] \nabla \Pi\|_{L^p} \lesssim d_j 2^{j(1-\frac{2}{p})} \|a\|_{\dot{B}_{p,1}^{\frac{2}{p}}} \|\nabla \Pi\|_{\dot{B}_{p,2}^{\frac{2}{p}-1}},$$

which along with (3.10) gives (3.3). \square

Motivated by [3, 4, 12, 14], we shall use Lemma 3.1 and a duality argument to prove the following crucial elliptic estimates in two space dimensions.

Proposition 3.1. *Let $(p, q) \in (1, \infty) \times [1, \infty)$, $a \in \dot{B}_{q,1}^{\frac{2}{q}}(\mathbb{R}^2)$ with $1 + a \geq \kappa > 0$. Let $F = (F^1, F^2) \in \dot{B}_{p,1}^{\frac{2}{p}-1}(\mathbb{R}^2)$ and $\nabla \Pi \in \dot{B}_{p,1}^{\frac{2}{p}-1}(\mathbb{R}^2)$ solve (1.5). Then there hold*

(i) If $p \in (\frac{1+\sqrt{17}}{4}, 2)$ and $\frac{1}{p} - \frac{1}{q} \leq \frac{1}{2}$, or $p \in (2, \frac{5+\sqrt{17}}{2})$ and $\frac{1}{p} + \frac{1}{q} \geq \frac{1}{2}$, then

$$\|\nabla \Pi\|_{\dot{B}_{p,2}^{\frac{2}{p}-1}} \lesssim (1 + \|a\|_{\dot{B}_{q,1}^{\frac{2}{q}}}) \|\mathcal{Q}F\|_{\dot{B}_{p,2}^{\frac{2}{p}-1}}. \quad (3.11)$$

(ii) If $p \in (1, 4)$ and $a \in \dot{B}_{p,1}^{\frac{2}{p}}(\mathbb{R}^2)$, then

$$\|\nabla \Pi\|_{\dot{B}_{p,1}^{\frac{2}{p}-1}} \lesssim (1 + \|a\|_{\dot{B}_{p,1}^{\frac{2}{p}}})^k \|\mathcal{Q}F\|_{\dot{B}_{p,1}^{\frac{2}{p}-1}}, \quad (3.12)$$

where $k = 1$ if $p \in (1, 2]$, and $k = 2$ if $p \in (2, 4)$.

Proof. For simplicity, we just present *a priori* estimate for smooth enough functions a , F and $\nabla \Pi$. Density arguments make the following computations rigorous. Thanks to $1 + a \geq \kappa > 0$ and $\operatorname{div} F = \operatorname{div} \mathcal{Q}F$, we readily deduce from (1.5) that

$$\kappa \|\nabla \Pi\|_{L^2} \leq \|\mathcal{Q}F\|_{L^2}. \quad (3.13)$$

Applying $\dot{\Delta}_j$ to (1.5) gives

$$\operatorname{div} ((1 + a)\dot{\Delta}_j \nabla \Pi) = \operatorname{div} \dot{\Delta}_j F - \operatorname{div} ([\dot{\Delta}_j, a]\nabla \Pi).$$

We next multiply the above equation by $-|\dot{\Delta}_j \Pi|^{p-2} \dot{\Delta}_j \Pi$ and integrate over \mathbb{R}^2 . Then applying Lemma 8 in Appendix B of [14] implies for some constants c and C

$$c\kappa 2^{2j} \|\dot{\Delta}_j \Pi\|_{L^p}^p \leq C 2^j \|\dot{\Delta}_j \mathcal{Q}F\|_{L^p} \|\dot{\Delta}_j \Pi\|_{L^p}^{p-1} + \int_{\mathbb{R}^2} \operatorname{div} ([\dot{\Delta}_j, a]\nabla \Pi) \cdot |\dot{\Delta}_j \Pi|^{p-2} \dot{\Delta}_j \Pi dx. \quad (3.14)$$

(i) If $p \in (\frac{1+\sqrt{17}}{4}, 2)$ with $\frac{1}{p} - \frac{1}{q} \leq \frac{1}{2}$, substituting (3.1) into (3.14) leads to

$$\|\dot{\Delta}_j \nabla \Pi\|_{L^p} \lesssim \|\dot{\Delta}_j \mathcal{Q}F\|_{L^p} + d_j 2^{j(1-\frac{2}{p})} \|a\|_{\dot{B}_{q,1}^{\frac{2}{q}}} \|\nabla \Pi\|_{L^2},$$

which along with (3.13) and the embedding $\dot{B}_{p,2}^{\frac{2}{p}-1}(\mathbb{R}^2) \hookrightarrow L^2(\mathbb{R}^2)$, $l^1 \hookrightarrow l^2$ gives

$$\|\nabla \Pi\|_{\dot{B}_{p,2}^{\frac{2}{p}-1}} \lesssim (1 + \|a\|_{\dot{B}_{q,1}^{\frac{2}{q}}}) \|\mathcal{Q}F\|_{\dot{B}_{p,2}^{\frac{2}{p}-1}}. \quad (3.15)$$

Next, we consider the case when $p \in (2, \frac{5+\sqrt{17}}{2})$ with $\frac{1}{p} + \frac{1}{q} \geq \frac{1}{2}$. In this case, motivated by [3, 4, 12], we shall use a duality argument:

$$\|\nabla \Pi\|_{\dot{B}_{p,2}^{\frac{2}{p}-1}} = \sup_{\|g\|_{\dot{B}_{p',2}^{1-\frac{2}{p}}}=1} \langle \nabla \Pi, g \rangle = \sup_{\|g\|_{\dot{B}_{p',2}^{\frac{2}{p'}-1}}=1} -\langle \Pi, \operatorname{div} g \rangle, \quad (3.16)$$

where $\langle \cdot, \cdot \rangle$ denotes the duality bracket between $\mathcal{S}'(\mathbb{R}^2)$ and $\mathcal{S}(\mathbb{R}^2)$. Notice that $p' \in (\frac{1+\sqrt{17}}{4}, 2)$ and $\frac{1}{p'} - \frac{1}{q} \leq \frac{1}{2}$, then applying (3.15) ensures that for any $g \in \dot{B}_{p',2}^{\frac{2}{p'}-1}(\mathbb{R}^2)$, there exists a unique solution $\nabla P_g \in \dot{B}_{p',2}^{\frac{2}{p'}-1}(\mathbb{R}^2)$ to the elliptic equation

$$\operatorname{div} ((1 + a)\nabla P_g) = \operatorname{div} g,$$

such that

$$\|\nabla P_g\|_{\dot{B}_{p',2}^{\frac{2}{p'}-1}} \lesssim (1 + \|a\|_{\dot{B}_{q,1}^{\frac{2}{q}}}) \|g\|_{\dot{B}_{p',2}^{\frac{2}{p'}-1}}. \quad (3.17)$$

We proceed

$$\begin{aligned} -\langle \Pi, \operatorname{div} g \rangle &= -\langle \Pi, \operatorname{div} ((1+a)\nabla P_g) \rangle = -\langle \operatorname{div} ((1+a)\nabla \Pi), P_g \rangle \\ &= -\langle \operatorname{div} F, P_g \rangle = \langle \mathcal{Q}F, \nabla P_g \rangle \leq \|\mathcal{Q}F\|_{\dot{B}_{p,2}^{\frac{2}{p}-1}} \|\nabla P_g\|_{\dot{B}_{p',2}^{\frac{2}{p'}-1}}, \end{aligned}$$

which along with (3.16) and (3.17) implies (3.11).

(ii) If $p \in (1, 2]$, substituting (3.2) into (3.14) and using a similar argument as (3.15) gives

$$\|\nabla \Pi\|_{\dot{B}_{p,1}^{\frac{2}{p}-1}} \lesssim (1 + \|a\|_{\dot{B}_{p,1}^{\frac{2}{p}}}) \|\mathcal{Q}F\|_{\dot{B}_{p,1}^{\frac{2}{p}-1}}.$$

If $p \in (2, 4)$, we get by substituting (3.3) into (3.14) that

$$\|\dot{\Delta}_j \nabla \Pi\|_{L^p} \lesssim \|\dot{\Delta}_j \mathcal{Q}F\|_{L^p} + d_j 2^{j(1-\frac{2}{p})} \|a\|_{\dot{B}_{p,1}^{\frac{2}{p}}} \|\nabla \Pi\|_{\dot{B}_{p,2}^{\frac{2}{p}-1}},$$

which along with (3.11) leads to

$$\|\nabla \Pi\|_{\dot{B}_{p,1}^{\frac{2}{p}-1}} \lesssim (1 + \|a\|_{\dot{B}_{p,1}^{\frac{2}{p}}})^2 \|\mathcal{Q}F\|_{\dot{B}_{p,1}^{\frac{2}{p}-1}}.$$

This completes the proof of the proposition. \square

Proposition 3.2. *Let $\mu > 0$, $1 < p < 4$, $u_0 \in \dot{B}_{p,1}^{\frac{2}{p}-1}(\mathbb{R}^2)$ and $a, b \in L_T^\infty(\dot{B}_{p,1}^{\frac{2}{p}}(\mathbb{R}^2))$ with $1 + a \geq \kappa > 0$ and $\mu + b \geq \kappa > 0$. Let $f \in L_T^1(\dot{B}_{p,1}^{\frac{2}{p}-1}(\mathbb{R}^2))$, $g \in L_T^1(\dot{B}_{p,1}^{\frac{2}{p}}(\mathbb{R}^2))$ and $\partial_t g = \operatorname{div} R$ with $R \in L_T^1(\dot{B}_{p,1}^{\frac{2}{p}-1}(\mathbb{R}^2))$. Let $(u, \nabla \Pi) \in C([0, T]; \dot{B}_{p,1}^{\frac{2}{p}-1}(\mathbb{R}^2)) \cap L_T^1(\dot{B}_{p,1}^{\frac{2}{p}+1}(\mathbb{R}^2)) \times L_T^1(\dot{B}_{p,1}^{\frac{2}{p}-1}(\mathbb{R}^2))$ solve*

$$\begin{cases} \partial_t u - \operatorname{div}((\mu + b)\nabla u) + (1 + a)\nabla \Pi = f, \\ \operatorname{div} u = g, \\ u|_{t=0} = u_0. \end{cases} \quad (3.18)$$

Then there holds for $t \in [0, T]$

$$\begin{aligned} &\|u\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{2}{p}-1})} + \|u\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}+1})} + \|\nabla \Pi\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} \\ &\lesssim \|u_0\|_{\dot{B}_{p,1}^{\frac{2}{p}-1}} + (1 + \|a\|_{L_t^\infty(\dot{B}_{p,1}^{\frac{2}{p}})})^3 \{ \|f\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} + \|R\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} \\ &\quad + (1 + \|b\|_{L_t^\infty(L^\infty)}) \|g\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}})} + 2^m \|b\|_{L_t^\infty(\dot{B}_{p,1}^{\frac{2}{p}})} \|u\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}})} \}, \end{aligned} \quad (3.19)$$

provided that

$$(1 + \|a\|_{L_T^\infty(\dot{B}_{p,1}^{\frac{2}{p}})})^3 \|b - \dot{S}_m b\|_{L_T^\infty(\dot{B}_{p,1}^{\frac{2}{p}})} \leq c_0 \quad (3.20)$$

for some sufficiently small positive constant c_0 and some integer $m \in \mathbb{Z}$.

Proof. Motivated by [3, 4, 13], we first use the decomposition $\text{Id} = \dot{S}_m + (\text{Id} - \dot{S}_m)$ to turn the u equation of (3.18) into

$$\partial_t u - \text{div}((\mu + \dot{S}_m b) \nabla u) + (1 + a) \nabla \Pi = f + \text{div}((b - \dot{S}_m b) \nabla u). \quad (3.21)$$

Thanks to $\mu + b \geq \kappa > 0$ and (3.20), we infer

$$\mu + \dot{S}_m b = \mu + b + (\dot{S}_m b - b) \geq \frac{1}{2} \kappa. \quad (3.22)$$

Then taking div to (3.21) and using $\text{div} u = g$ and $\partial_t g = \text{div} R$ leads to

$$\text{div}((1 + a) \nabla \Pi) = \text{div}(f + \text{div}((b - \dot{S}_m b) \nabla u) - R + \mu \nabla g + \nabla \dot{S}_m b \cdot \nabla u + \dot{S}_m b \Delta u),$$

from which, we get by applying (3.12) that

$$\begin{aligned} \|\nabla \Pi\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} &\lesssim (1 + \|a\|_{L_t^\infty(\dot{B}_{p,1}^{\frac{2}{p}})})^2 \{ \|f\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} + \|(b - \dot{S}_m b) \nabla u\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}})} \\ &\quad + \|R\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} + \|g\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}})} + \|\nabla \dot{S}_m b \cdot \nabla u\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} \\ &\quad + \|\mathcal{Q}(\dot{S}_m b \Delta u)\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} \}. \end{aligned} \quad (3.23)$$

Using product laws in Besov spaces, we get

$$\begin{aligned} &\|(b - \dot{S}_m b) \nabla u\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}})} + \|\nabla \dot{S}_m b \cdot \nabla u\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} \\ &\lesssim \|b - \dot{S}_m b\|_{L_t^\infty(\dot{B}_{p,1}^{\frac{2}{p}})} \|u\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}+1})} + 2^m \|b\|_{L_t^\infty(\dot{B}_{p,1}^{\frac{2}{p}})} \|u\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}})}. \end{aligned} \quad (3.24)$$

Yet notice that $\mathcal{Q} = -\nabla(-\Delta)^{-1} \text{div}$ and $\text{div} u = g$, we get by applying Bony's decomposition

$$\mathcal{Q}(\dot{S}_m b \Delta u) = -\nabla(-\Delta)^{-1} (\dot{T}_{\nabla \dot{S}_m b} \Delta u + \dot{T}_{\dot{S}_m b} \Delta g) + \mathcal{Q}(\dot{T}_{\Delta u} \dot{S}_m b) + \mathcal{Q}(\dot{R}(\dot{S}_m b, \Delta u)).$$

Then it is easy to get

$$\begin{aligned} \|\dot{\Delta}_j(\dot{T}_{\nabla \dot{S}_m b} \Delta u)\|_{L^p} &\lesssim \sum_{|j'-j| \leq 4} \|\dot{S}_{j'-1} \nabla \dot{S}_m b\|_{L^\infty} \|\dot{\Delta}_{j'} \Delta u\|_{L^p} \lesssim d_j 2^{j(2-\frac{2}{p})+m} \|b\|_{L^\infty} \|u\|_{\dot{B}_{p,1}^{\frac{2}{p}}}, \\ \|\dot{\Delta}_j(\dot{T}_{\dot{S}_m b} \Delta g)\|_{L^p} &\lesssim \sum_{|j'-j| \leq 4} \|\dot{S}_{j'-1} \dot{S}_m b\|_{L^\infty} \|\dot{\Delta}_{j'} \Delta g\|_{L^p} \lesssim d_j 2^{j(2-\frac{2}{p})} \|b\|_{L^\infty} \|g\|_{\dot{B}_{p,1}^{\frac{2}{p}}}, \\ \|\dot{\Delta}_j(\dot{T}_{\Delta u} \dot{S}_m b)\|_{L^p} &\lesssim \sum_{|j'-j| \leq 4} \|\dot{\Delta}_{j'} \dot{S}_m b\|_{L^p} \|\dot{S}_{j'-1} \Delta u\|_{L^\infty} \lesssim d_j 2^{j(1-\frac{2}{p})+m} \|b\|_{\dot{B}_{p,1}^{\frac{2}{p}}} \|u\|_{\dot{B}_{p,1}^{\frac{2}{p}}}. \end{aligned}$$

While for $p \in [2, 4)$, applying Lemma 2.1 yields

$$\|\dot{\Delta}_j \dot{R}(\dot{S}_m b, \Delta u)\|_{L^p} \lesssim 2^{\frac{2j}{p}} \sum_{j' \geq j-3} \|\dot{\Delta}_{j'} \dot{S}_m b\|_{L^p} \|\tilde{\dot{\Delta}}_{j'} \Delta u\|_{L^p} \lesssim d_j 2^{j(1-\frac{2}{p})+m} \|b\|_{\dot{B}_{p,1}^{\frac{2}{p}}} \|u\|_{\dot{B}_{p,1}^{\frac{2}{p}}}.$$

And for $p \in (1, 2)$, we have $p' = \frac{p}{p-1} > p$ so that we could alternatively get

$$\|\dot{\Delta}_j \dot{R}(\dot{S}_m b, \Delta u)\|_{L^p} \lesssim 2^{\frac{2j}{p'}} \sum_{j' \geq j-3} \|\dot{\Delta}_{j'} \dot{S}_m b\|_{L^{p'}} \|\tilde{\dot{\Delta}}_{j'} \Delta u\|_{L^p} \lesssim d_j 2^{j(1-\frac{2}{p})+m} \|b\|_{\dot{B}_{p,1}^{\frac{2}{p}}} \|u\|_{\dot{B}_{p,1}^{\frac{2}{p}}}.$$

Whence we conclude that

$$\|\mathcal{Q}(\dot{S}_m b \Delta u)\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} \lesssim 2^m \|b\|_{L_t^\infty(\dot{B}_{p,1}^{\frac{2}{p}})} \|u\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}})} + \|b\|_{L_t^\infty(L^\infty)} \|g\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}})}. \quad (3.25)$$

Substituting (3.24) and (3.25) into (3.23), we arrive at

$$\begin{aligned} \|\nabla \Pi\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} &\lesssim (1 + \|a\|_{L_t^\infty(\dot{B}_{p,1}^{\frac{2}{p}})})^2 \{ \|f\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} + \|R\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} \\ &\quad + (1 + \|b\|_{L_t^\infty(L^\infty)}) \|g\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}})} + 2^m \|b\|_{L_t^\infty(\dot{B}_{p,1}^{\frac{2}{p}})} \|u\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}})} \\ &\quad + \|b - \dot{S}_m b\|_{L_t^\infty(\dot{B}_{p,1}^{\frac{2}{p}})} \|u\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}+1})} \}. \end{aligned} \quad (3.26)$$

On the other hand, applying $\dot{\Delta}_j$ to (3.21), we arrive at

$$\begin{aligned} &\partial_t \dot{\Delta}_j u - \operatorname{div}((\mu + \dot{S}_m b) \dot{\Delta}_j \nabla u) + \dot{\Delta}_j((1+a) \nabla \Pi) \\ &= \dot{\Delta}_j f + \dot{\Delta}_j \operatorname{div}((b - \dot{S}_m b) \nabla u) + \operatorname{div}([\dot{\Delta}_j, \dot{S}_m b] \nabla u). \end{aligned}$$

Multiplying the i -th ($i = 1, 2$) equation by $|\dot{\Delta}_j u^i|^{p-2} \dot{\Delta}_j u^i$ and using a similar argument as (3.14) leads to

$$\begin{aligned} \frac{d}{dt} \|\dot{\Delta}_j u\|_{L^p} + c\kappa 2^{2j} \|\dot{\Delta}_j u\|_{L^p} &\lesssim \|\dot{\Delta}_j f\|_{L^p} + 2^j \|\dot{\Delta}_j((b - \dot{S}_m b) \nabla u)\|_{L^p} \\ &\quad + \|\operatorname{div}([\dot{\Delta}_j, \dot{S}_m b] \nabla u)\|_{L^p} + \|\dot{\Delta}_j((1+a) \nabla \Pi)\|_{L^p}. \end{aligned}$$

After time integration, multiplying $2^{(\frac{2}{p}-1)j}$ and summing up over j , we get

$$\begin{aligned} &\|u\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{2}{p}-1})} + \|u\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}+1})} \\ &\lesssim \|u_0\|_{\dot{B}_{p,1}^{\frac{2}{p}-1}} + \|f\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} + \|b - \dot{S}_m b\|_{L_t^\infty(\dot{B}_{p,1}^{\frac{2}{p}})} \|u\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}+1})} \\ &\quad + 2^m \|b\|_{L_t^\infty(\dot{B}_{p,1}^{\frac{2}{p}})} \|u\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}})} + (1 + \|a\|_{L_t^\infty(\dot{B}_{p,1}^{\frac{2}{p}})}) \|\nabla \Pi\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})}. \end{aligned} \quad (3.27)$$

where we used product laws and Lemma 5 in the appendix of [15]

$$\sum_{j \in \mathbb{Z}} 2^{j(\frac{2}{p}-1)} \|\operatorname{div}([\dot{\Delta}_j, \dot{S}_m b] \nabla u)\|_{L_t^1(L^p)} \lesssim 2^m \|b\|_{L_t^\infty(\dot{B}_{p,1}^{\frac{2}{p}})} \|u\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}})}.$$

Combining (3.26) and (3.27) and using (3.20), we conclude the proof of (3.19). \square

The following corollary will be used to prove the local well-posedness part of Theorem 1.1.

Corollary 3.1. *Let p, u_0, a, f, g, R be given in Proposition 3.2 and $\|a\|_{L^\infty} \leq C_0$. Let $(u, \nabla \Pi) \in C([0, T]; \dot{B}_{p,1}^{\frac{2}{p}-1}(\mathbb{R}^2)) \cap L_T^1(\dot{B}_{p,1}^{\frac{2}{p}+1}(\mathbb{R}^2)) \times L_T^1(\dot{B}_{p,1}^{\frac{2}{p}-1}(\mathbb{R}^2))$ solve*

$$\begin{cases} \partial_t u - (1+a) \operatorname{div}(2\tilde{\mu}(a)\mathcal{M}(u)) + (1+a)\nabla \Pi = f, \\ \operatorname{div} u = g, \\ u|_{t=0} = u_0, \end{cases}$$

with some smooth, positive function $\tilde{\mu}(a)$. Further, denote by $b = b(a) \stackrel{\text{def}}{=} (1+a)\tilde{\mu}(a) - \tilde{\mu}(0)$ and $\lambda = \lambda(a) \stackrel{\text{def}}{=} \int_0^a \tilde{\mu}(s) ds$. If there exist some sufficiently small positive constant c_0 and some integer $m \in \mathbb{Z}$ such that

$$(1 + \|a\|_{L_T^\infty(\dot{B}_{p,1}^{\frac{2}{p}})})^3 \|b - \dot{S}_m b, \lambda - \dot{S}_m \lambda\|_{L_T^\infty(\dot{B}_{p,1}^{\frac{2}{p}})} \leq c_0, \quad (3.28)$$

then we have for $t \in [0, T]$

$$\begin{aligned} & \|u\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{2}{p}-1})} + \|u\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}+1})} + \|\nabla \Pi\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} \\ & \lesssim \|u_0\|_{\dot{B}_{p,1}^{\frac{2}{p}-1}} + (1 + \|a\|_{L_t^\infty(\dot{B}_{p,1}^{\frac{2}{p}})})^3 \{ \|f\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} + \|R\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} \\ & \quad + (1 + \|a\|_{L_t^\infty(\dot{B}_{p,1}^{\frac{2}{p}})}) \|g\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}})} + 2^m \|a\|_{L_t^\infty(\dot{B}_{p,1}^{\frac{2}{p}})} \|u\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}})} \}. \end{aligned} \quad (3.29)$$

Proof. Since $b(a)$ and $\lambda(a)$ are smooth functions with $b(0) = \lambda(0) = 0$, we get by applying Theorem 2.61 in [7] that

$$\|b, \lambda\|_{\dot{B}_{p,1}^{\frac{2}{p}}} \lesssim (1 + \|a\|_{L^\infty})^{[\frac{2}{p}]+1} \|b', \lambda'\|_{W^{[\frac{2}{p}]+1, \infty}} \|a\|_{\dot{B}_{p,1}^{\frac{2}{p}}} \lesssim \|a\|_{\dot{B}_{p,1}^{\frac{2}{p}}}. \quad (3.30)$$

Thanks to $\operatorname{div} u = g$, we rewrite the equation for u as follow:

$$\begin{aligned} & \partial_t u - \operatorname{div}((\tilde{\mu}(0) + b)\nabla u) + (1+a)\nabla \Pi \\ & = \tilde{f} \stackrel{\text{def}}{=} f + (\tilde{\mu}(0) + b)\nabla g + \nabla u \cdot \nabla b - 2\mathcal{M}(u) \cdot \nabla \lambda, \end{aligned} \quad (3.31)$$

where $(\nabla u \cdot \nabla b)_i \stackrel{\text{def}}{=} \partial_i u \cdot \nabla b = \partial_i u^j \partial_j b$. While applying (3.30) and product laws in Besov spaces gives rise to

$$\begin{aligned} \|\tilde{f}\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} &\lesssim \|f\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} + (1 + \|a\|_{L_t^\infty(\dot{B}_{p,1}^{\frac{2}{p}})}) \|g\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}})} + 2^m \|a\|_{L_t^\infty(\dot{B}_{p,1}^{\frac{2}{p}})} \|u\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}})} \\ &\quad + \|b - \dot{S}_m a, \lambda - \dot{S}_m \lambda\|_{L_t^\infty(\dot{B}_{p,1}^{\frac{2}{p}})} \|u\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}+1})}, \end{aligned}$$

from which and (3.28), we apply Proposition 3.2 to (3.31) to conclude the proof of (3.29). \square

4. Local well-posedness of (1.2)

4.1. Local existence

Step 1. Construction of smooth approximate solutions.

Firstly, there exists a sequence $\{(a_0^n, \tilde{u}_0^n)\}_{n \in \mathbb{N}} \subset \mathcal{S}(\mathbb{R}^2)$ such that (a_0^n, \tilde{u}_0^n) converges to (a_0, u_0) in $\dot{B}_{p,1}^{\frac{2}{p}}(\mathbb{R}^2) \times \dot{B}_{p,1}^{\frac{2}{p}-1}(\mathbb{R}^2)$. Define $u_0^n \stackrel{\text{def}}{=} \mathbb{P} \tilde{u}_0^n$ so that $\text{div } u_0^n = 0$. Then u_0^n belongs to $H^\infty(\mathbb{R}^2)$ and converges to u_0 in $\dot{B}_{p,1}^{\frac{2}{p}-1}(\mathbb{R}^2)$. Furthermore, we could assume that

$$\begin{aligned} \|a_0^n\|_{L^\infty} &\leq 2\|a_0\|_{L^\infty}, \quad \|a_0^n\|_{\dot{B}_{p,1}^{\frac{2}{p}}} \leq 2\|a_0\|_{\dot{B}_{p,1}^{\frac{2}{p}}}, \quad \|u_0^n\|_{\dot{B}_{p,1}^{\frac{2}{p}-1}} \leq 2\|u_0\|_{\dot{B}_{p,1}^{\frac{2}{p}-1}}, \\ \text{and } 1 + a_0^n &= 1 + a_0 + (a_0^n - a_0) \geq \frac{1}{2}\kappa. \end{aligned}$$

Whence applying Theorem 1.2 of [1] ensures that the system (1.2) with the initial data (a_0^n, u_0^n) admits a unique local in time solution $(a^n, u^n, \nabla \Pi^n)$ belonging to

$$\begin{aligned} &C([0, T^n]; H^{\alpha+1}(\mathbb{R}^2)) \times (C([0, T^n]; H^\alpha(\mathbb{R}^2)) \cap \tilde{L}_{loc}^1(0, T^n; H^{\alpha+2}(\mathbb{R}^2))) \\ &\quad \times \tilde{L}_{loc}^1(0, T^n; H^\alpha(\mathbb{R}^2)) \quad \text{with } \alpha > 0. \end{aligned}$$

Moreover, we deduce from the transport equation of (1.2) that

$$\begin{aligned} 1 + \inf_{(t,x) \in [0, T^n] \times \mathbb{R}^2} a^n(t, x) &= 1 + \inf_{y \in \mathbb{R}^2} a_0^n(y) \geq \frac{1}{2}\kappa, \\ \|a^n(t)\|_{L^\infty} &= \|a_0^n\|_{L^\infty} \leq 2\|a_0\|_{L^\infty}, \quad \forall t \in [0, T^n]. \end{aligned} \tag{4.1}$$

Step 2. Uniform estimates to the approximate solutions.

Next, we shall prove that there exists a positive time $T < \inf_{n \in \mathbb{N}} T^n$ such that $(a^n, u^n, \nabla \Pi^n)$ is uniformly bounded in the space

$$E_T \stackrel{\text{def}}{=} \tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{2}{p}}) \times (\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{2}{p}-1}) \cap L_T^1(\dot{B}_{p,1}^{\frac{2}{p}+1})) \times L_T^1(\dot{B}_{p,1}^{\frac{2}{p}-1}).$$

For this, let $(u_L(t), u_L^n(t)) \stackrel{\text{def}}{=} e^{\mu t \Delta} (u_0, u_0^n)$ with $\mu \stackrel{\text{def}}{=} \tilde{\mu}(0)$. Then it is easy to observe that

$$\|u_L^n\|_{\tilde{L}^\infty(\mathbb{R}^+; \dot{B}_{p,1}^{\frac{2}{p}-1})} + \mu \|u_L^n\|_{L^1(\mathbb{R}^+; \dot{B}_{p,1}^{\frac{2}{p}+1})} \lesssim \|u_0^n\|_{\dot{B}_{p,1}^{\frac{2}{p}-1}} \lesssim \|u_0\|_{\dot{B}_{p,1}^{\frac{2}{p}-1}}, \quad (4.2)$$

and

$$\|u_L^n\|_{L_T^1(\dot{B}_{p,1}^{\frac{2}{p}+1})} \leq \|u_L\|_{L_T^1(\dot{B}_{p,1}^{\frac{2}{p}+1})} + C \|u_0^n - u_0\|_{\dot{B}_{p,1}^{\frac{2}{p}-1}}.$$

Whence for any $\varepsilon > 0$, there exist a number $k = k(\varepsilon) \in \mathbb{N}$ and a positive time $T = T(\varepsilon, u_0)$ such that

$$\sup_{n \geq k} \|u_L^n\|_{L_T^1(\dot{B}_{p,1}^{\frac{2}{p}+1})} \leq \varepsilon. \quad (4.3)$$

Denote by $\bar{u}^n \stackrel{\text{def}}{=} u^n - u_L^n$. Then the system for $(a^n, \bar{u}^n, \nabla \Pi^n)$ reads

$$\begin{cases} \partial_t a^n + (u_L^n + \bar{u}^n) \cdot \nabla a^n = 0, \\ \partial_t \bar{u}^n - (1 + a^n) \operatorname{div} (2\tilde{\mu}(a^n) \mathcal{M}(\bar{u}^n)) + (1 + a^n) \nabla \Pi^n = F_n, \\ \operatorname{div} \bar{u}^n = 0, \\ (a^n, \bar{u}^n)|_{t=0} = (a_0^n, 0), \end{cases} \quad (4.4)$$

where

$$\begin{aligned} F_n &= -\bar{u}^n \cdot \nabla \bar{u}^n - u_L^n \cdot \nabla u_L^n - \operatorname{div}(\bar{u}^n \otimes u_L^n + u_L^n \otimes \bar{u}^n) \\ &\quad + \operatorname{div} (2(\tilde{\mu}(a^n) - \tilde{\mu}(0)) \mathcal{M}(u_L^n)) + a^n \operatorname{div} (2\tilde{\mu}(a^n) \mathcal{M}(u_L^n)). \end{aligned}$$

For notational simplicity, we denote by $A^n(t) \stackrel{\text{def}}{=} \|a^n\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{2}{p}})}$ and

$$Z^n(t) \stackrel{\text{def}}{=} \|\bar{u}^n\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{2}{p}-1})} + \|\bar{u}^n\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}+1})} + \|\nabla \Pi^n\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})}.$$

Then thanks to (4.2), we get by applying product laws and interpolation inequality in Besov spaces that

$$\begin{aligned} \|\bar{u}^n \cdot \nabla \bar{u}^n + u_L^n \cdot \nabla u_L^n\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} &\lesssim (Z^n(t))^2 + \|u_0\|_{\dot{B}_{p,1}^{\frac{2}{p}-1}} \|u_L^n\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}+1})}, \\ \|\operatorname{div}(\bar{u}^n \otimes u_L^n + u_L^n \otimes \bar{u}^n)\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} &\lesssim \|u_L^n\|_{L_t^2(\dot{B}_{p,1}^{\frac{2}{p}})} Z^n(t). \end{aligned}$$

Along the same line, one has

$$\begin{aligned}\|\operatorname{div} (2(\tilde{\mu}(a^n) - \tilde{\mu}(0))\mathcal{M}(u_L^n))\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} &\lesssim A^n(t)\|u_L^n\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}+1})}, \\ \|a^n \operatorname{div} (2\tilde{\mu}(a^n)\mathcal{M}(u_L^n))\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} &\lesssim A^n(t)(1 + A^n(t))\|u_L^n\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}+1})}.\end{aligned}$$

As a consequence, we obtain

$$\begin{aligned}\|F_n\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} &\lesssim (Z^n(t) + \|u_L^n\|_{L_t^2(\dot{B}_{p,1}^{\frac{2}{p}})})Z^n(t) \\ &\quad + (\|u_0\|_{\dot{B}_{p,1}^{\frac{2}{p}-1}} + A^n(t)(1 + A^n(t)))\|u_L^n\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}+1})}.\end{aligned}\tag{4.5}$$

Denote by $b^n \stackrel{\text{def}}{=} (1 + a^n)\tilde{\mu}(a^n) - \tilde{\mu}(0)$, $\lambda^n \stackrel{\text{def}}{=} \int_0^{a^n} \tilde{\mu}(s)ds$. Applying Corollary 3.1 to the \bar{u}^n equation of (4.4), we get for $t \in [0, T] \subset [0, T^n)$ that

$$Z^n(t) \lesssim (1 + A^n(t))^3 (\|F_n\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} + 2^m A^n(t)\|\bar{u}^n\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}})}),\tag{4.6}$$

provided that

$$(1 + A^n(T))^3 \|b^n - \dot{S}_m b^n, \lambda^n - \dot{S}_m \lambda^n\|_{L_T^\infty(\dot{B}_{p,1}^{\frac{2}{p}})} \leq c_0\tag{4.7}$$

for some sufficiently small positive constant c_0 and some integer $m \in \mathbb{Z}$. Substituting (4.5) into (4.6) and using interpolation, we arrive at

$$\begin{aligned}Z^n(t) &\leq C(1 + A^n(t))^4 \{ (Z^n(t) + \|u_L^n\|_{L_t^2(\dot{B}_{p,1}^{\frac{2}{p}})} + 2^m t^{\frac{1}{2}})Z^n(t) \\ &\quad + (\|u_0\|_{\dot{B}_{p,1}^{\frac{2}{p}-1}} + A^n(t))\|u_L^n\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}+1})} \}.\end{aligned}\tag{4.8}$$

On the other hand, applying (2.1) to the transport equation of (4.4), we have for $t \in [0, T^n)$

$$A^n(t) \leq C\|a_0\|_{\dot{B}_{p,1}^{\frac{2}{p}}} \exp \left(C(Z^n(t) + \|u_0\|_{\dot{B}_{p,1}^{\frac{2}{p}-1}}) \right).\tag{4.9}$$

However, for any function $\chi \in \mathcal{D}(\mathbb{R})$ vanishing at 0, the composite function $\chi(a^n)$ with initial data $\chi(a_0^n)$ also solves the renormalized transport equation

$$\partial_t \chi(a^n) + (u_L^n + \bar{u}^n) \cdot \nabla \chi(a^n) = 0.$$

Then applying (2.2) to the above equation gives rise to

$$\begin{aligned}
& \|\chi(a^n) - \dot{S}_m \chi(a^n)\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{2}{p}})} \\
& \leq \sum_{j \geq m} 2^{\frac{2j}{p}} \|\dot{\Delta}_j \chi(a_0^n)\|_{L^p} + \|\chi(a_0^n)\|_{\dot{B}_{p,1}^{\frac{2}{p}}} \left(\exp(CZ^n(t) + C\|u_L^n\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}+1})}) - 1 \right) \\
& \leq \sum_{j \geq m} 2^{\frac{2j}{p}} \|\dot{\Delta}_j \chi(a_0)\|_{L^p} + C(1 + \|a_0\|_{\dot{B}_{p,1}^{\frac{2}{p}}}) \|a_0^n - a_0\|_{\dot{B}_{p,1}^{\frac{2}{p}}} \\
& \quad + C\|a_0\|_{\dot{B}_{p,1}^{\frac{2}{p}}} \left(\exp(CZ^n(t) + C\|u_L^n\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}+1})}) - 1 \right),
\end{aligned}$$

where we used

$$\begin{aligned}
\|\chi(a_0^n) - \chi(a_0)\|_{\dot{B}_{p,1}^{\frac{2}{p}}} &= \left\| (a_0^n - a_0) \int_0^1 \chi'(\tau a_0^n + (1-\tau)a_0) d\tau \right\|_{\dot{B}_{p,1}^{\frac{2}{p}}} \\
&\leq C(1 + \|a_0\|_{\dot{B}_{p,1}^{\frac{2}{p}}}) \|a_0^n - a_0\|_{\dot{B}_{p,1}^{\frac{2}{p}}}.
\end{aligned}$$

As a consequence, we obtain for $t \in [0, T^n]$

$$\begin{aligned}
& \|b^n - \dot{S}_m b^n, \lambda^n - \dot{S}_m \lambda^n\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{2}{p}})} \\
& \leq \sum_{j \geq m} 2^{\frac{2j}{p}} \|\dot{\Delta}_j b_0, \dot{\Delta}_j \lambda_0\|_{L^p} + C(1 + \|a_0\|_{\dot{B}_{p,1}^{\frac{2}{p}}}) \|a_0^n - a_0\|_{\dot{B}_{p,1}^{\frac{2}{p}}} \\
& \quad + C\|a_0\|_{\dot{B}_{p,1}^{\frac{2}{p}}} \left(\exp(CZ^n(t) + C\|u_L^n\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}+1})}) - 1 \right), \tag{4.10}
\end{aligned}$$

with $b_0 \stackrel{\text{def}}{=} (1 + a_0)\tilde{\mu}(a_0) - \tilde{\mu}(0)$, $\lambda_0 \stackrel{\text{def}}{=} \int_0^{a_0} \tilde{\mu}(s) ds$.

Next, for any $n \in \mathbb{N}$, we define

$$T_*^n \stackrel{\text{def}}{=} \sup\{t \in (0, T^n) : Z^n(t) \leq 2\varepsilon_0\} \tag{4.11}$$

with $\varepsilon_0 \in (0, \frac{1}{2})$ to be determined. We shall prove $\inf_{n \in \mathbb{N}} T_*^n > 0$.

Firstly, we deduce from (4.9) and (4.11) for $t \leq T_*^n$ that

$$A^n(t) \leq C\|a_0\|_{\dot{B}_{p,1}^{\frac{2}{p}}} \exp\left(C(1 + \|u_0\|_{\dot{B}_{p,1}^{\frac{2}{p}-1}})\right) \stackrel{\text{def}}{=} A_0. \tag{4.12}$$

Notice that $(b_0, \lambda_0) \in (\dot{B}_{p,1}^{\frac{2}{p}}(\mathbb{R}^2))^2$, there exist $m = m(c_0) \in \mathbb{Z}$ and $n_0 = n_0(c_0) \in \mathbb{N}$ such that

$$(1 + A_0)^3 \left(\sum_{j \geq m} 2^{\frac{2j}{p}} \|\dot{\Delta}_j b_0, \dot{\Delta}_j \lambda_0\|_{L^p} + C(1 + \|a_0\|_{\dot{B}_{p,1}^{\frac{2}{p}}}) \sup_{n \geq n_0} \|a_0^n - a_0\|_{\dot{B}_{p,1}^{\frac{2}{p}}} \right) \leq \frac{1}{2} c_0. \tag{4.13}$$

Yet thanks to (4.3), taking ε_0 and T_0 small enough and $n_1 \geq n_0$ large enough ensures

$$C(1 + A_0)^3 \|a_0\|_{\dot{B}_{p,1}^{\frac{2}{p}}} \left(\exp \left(2C\varepsilon_0 + C \sup_{n \geq n_1} \|u_L^n\|_{L_{T_0}^1(\dot{B}_{p,1}^{\frac{2}{p}+1})} \right) - 1 \right) \leq \frac{1}{2}c_0. \quad (4.14)$$

Combining (4.12)–(4.14) implies that (4.7) with $T = \min\{T_*, T_0\}$ is fulfilled for any $n \geq n_1$.

Without loss of generality, we may assume that $T_* \leq T_0$. Then for any $t \leq T_*$, we deduce from (4.8) that

$$\begin{aligned} Z^n(t) \leq & C(1 + A_0)^4 \left\{ (2\varepsilon_0 + \|u_L^n\|_{L_t^2(\dot{B}_{p,1}^{\frac{2}{p}})} + 2^m t^{\frac{1}{2}}) Z^n(t) \right. \\ & \left. + (\|u_0\|_{\dot{B}_{p,1}^{\frac{2}{p}-1}} + A_0) \|u_L^n\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}+1})} \right\}. \end{aligned} \quad (4.15)$$

Finally, taking ε_0 and T_1 small enough and $n_2 \geq n_1$ large enough ensures for any $n \geq n_2$

$$C(1 + A_0)^4 (2\varepsilon_0 + \|u_L^n\|_{L_{T_1}^2(\dot{B}_{p,1}^{\frac{2}{p}})} + 2^m T_1^{\frac{1}{2}}) \leq \frac{1}{2},$$

and

$$2C(1 + A_0)^4 (\|u_0\|_{\dot{B}_{p,1}^{\frac{2}{p}-1}} + A_0) \|u_L^n\|_{L_{T_1}^1(\dot{B}_{p,1}^{\frac{2}{p}+1})} \leq \varepsilon_0,$$

which together with (4.15) implies

$$Z^n(t) \leq \varepsilon_0, \quad \forall t \leq \min(T_*, T_1), \quad n \geq n_2.$$

However, by the definition of T_* , we eventually conclude $T_* \geq T_1$ and $\sup_{n \geq n_2} Z^n(T_1) \leq \varepsilon_0$, which along with (4.2) and (4.12) ensures that

$$(a^n, u^n, \nabla \Pi^n) \text{ is uniformly bounded in } E_{T_1}. \quad (4.16)$$

Step 3. Convergence.

Thanks to (4.16), we can repeat the compactness argument in Step 3 to the proof of Theorem 5.1 in [10] to conclude that $(a^n, u^n, \nabla \Pi^n)$ converges to some limit $(a, u, \nabla \Pi)$ which satisfies (1.3) and solves the system (1.2) on $[0, T_1]$. Moreover, there exist some sufficiently small positive constant c_0 and some integer $m \in \mathbb{Z}$ such that

$$\|a - \dot{S}_m a\|_{L_{T_1}^\infty(\dot{B}_{p,1}^{\frac{2}{p}})} \leq c_0. \quad (4.17)$$

4.2. Uniqueness

As in [15–17], we apply Lagrangian approach to prove the uniqueness part of Theorem 1.1.

Before going further, we give some notations. For a matrix $A = (A^{ij}) : \mathbb{R}^2 \rightarrow \mathbb{R}^{2 \times 2}$, we denote A^T its transpose matrix, $\text{Tr}(A)$ its trace, $\det(A)$ its determinant and $(\text{div } A)^j \stackrel{\text{def}}{=} \partial_1 A^{1j} + \partial_2 A^{2j}$. For a C^1 vector field $u : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, denote $(Du)^{ij} \stackrel{\text{def}}{=} \partial_j u^i$, $\nabla u \stackrel{\text{def}}{=} (Du)^T$ and $\mathcal{M}_A(u) \stackrel{\text{def}}{=} \frac{1}{2}(Du \cdot A + A^T \cdot \nabla u)$.

Let $(a, u, \nabla \Pi)$ be a solution to (1.2) on $[0, T]$ and satisfy (1.3). By virtue of Cauchy-Lipschitz theorem, the unique trajectory $X(t, y)$ of u is determined by

$$X(t, y) = y + \int_0^t u(\tau, X(\tau, y)) d\tau, \quad t \in [0, T]$$

such that $X(t, \cdot)$ is a C^1 -diffeomorphism over \mathbb{R}^2 . Denote $A(t, y) \stackrel{\text{def}}{=} (D_y X(t, y))^{-1}$. Then the divergence free condition for u is equivalent to $\det(A) \equiv 1$. To obtain the Lagrangian formulation of (1.2), we define

$$(\eta, v, P)(t, y) \stackrel{\text{def}}{=} (a, u, \Pi)(t, X(t, y)). \quad (4.18)$$

If $\|\nabla u\|_{L_T^1(\dot{B}_{p,1}^{\frac{2}{p}})}$ is sufficiently small, then applying Proposition 8 in the appendix of [15] implies that $(\eta, v, \nabla P)$ belongs to the same functional space as $(a, u, \nabla \Pi)$. On the other hand, using the chain rule, we easily deduce that

$$\begin{aligned} (\partial_t a + u \cdot \nabla_x a)(t, X(t, y)) &= \partial_t \eta(t, y), \\ (\partial_t u + u \cdot \nabla_x u)(t, X(t, y)) &= \partial_t v(t, y), \\ \nabla_x \Pi(t, X(t, y)) &= A^T \nabla_y P(t, y). \end{aligned}$$

While applying Lemma A.1 in the appendix of [16] gives

$$\text{div}_x u(t, X(t, y)) = \text{Tr}(D_y v \cdot A)(t, y) = \text{div}_y (Av)(t, y), \quad (4.19)$$

and

$$\text{div}_x (\tilde{\mu}(a) \mathcal{M}(u))(t, X(t, y)) = \text{div}_y (\tilde{\mu}(a_0) A \mathcal{M}(v))(t, y).$$

Whence we deduce from (1.2) that $\eta(t, \cdot) \equiv a_0$ and $(v, \nabla P)$ solves

$$\begin{cases} \partial_t v - (1 + a_0) \operatorname{div}(2\tilde{\mu}(a_0) A \mathcal{M}_A(v)) + (1 + a_0) A^T \nabla P = 0, \\ \operatorname{div}(Av) = 0, \\ v|_{t=0} = u_0. \end{cases}$$

Now let $(a_i, u_i, \nabla \Pi_i)$ ($i = 1, 2$) be two solutions to (1.2) and (η_i, v_i, P_i) be determined by (4.18). Denote $(\delta v, \delta P, \delta A) \stackrel{\text{def}}{=} (v_2 - v_1, P_2 - P_1, A_2 - A_1)$, where

$$A_i(t, y) \stackrel{\text{def}}{=} \left(\operatorname{Id} + \int_0^t Dv_i(\tau, y) d\tau \right)^{-1}, \quad \text{for } i = 1, 2.$$

Then the system for $(\delta v, \nabla \delta P)$ reads

$$\begin{cases} \partial_t \delta v - (1 + a_0) \operatorname{div}(2\tilde{\mu}(a_0) \mathcal{M}(\delta v)) + (1 + a_0) \nabla \delta P = (1 + a_0) \delta F, \\ \operatorname{div} \delta v = g, \\ \delta v|_{t=0} = 0, \end{cases} \quad (4.20)$$

where $g \stackrel{\text{def}}{=} \operatorname{div}((\operatorname{Id} - A_2) \delta v - \delta A v_1)$, $R \stackrel{\text{def}}{=} -\partial_t A_2 \delta v + (\operatorname{Id} - A_2) \partial_t \delta v - \partial_t \delta A v_1 - \delta A \partial_t v_1$, and $\delta F \stackrel{\text{def}}{=} \sum_{k=1}^6 \delta F_k$ with

$$\begin{aligned} \delta F_1 &\stackrel{\text{def}}{=} (\operatorname{Id} - A_2^T) \nabla \delta P, & \delta F_2 &\stackrel{\text{def}}{=} -\delta A^T \nabla P_1, \\ \delta F_3 &\stackrel{\text{def}}{=} \operatorname{div}(2\tilde{\mu}(a_0) (A_2 - \operatorname{Id}) \mathcal{M}(\delta v)), & \delta F_4 &\stackrel{\text{def}}{=} \operatorname{div}(2\tilde{\mu}(a_0) A_2 \mathcal{M}_{A_2 - \operatorname{Id}}(\delta v)), \\ \delta F_5 &\stackrel{\text{def}}{=} \operatorname{div}(2\tilde{\mu}(a_0) A_2 \mathcal{M}_{\delta A}(v_1)), & \delta F_6 &\stackrel{\text{def}}{=} \operatorname{div}(2\tilde{\mu}(a_0) \delta A \mathcal{M}_{A_1}(v_1)). \end{aligned}$$

In the sequel, we shall take T to be so small that

$$\int_0^T \|Dv_i(t)\|_{\dot{B}_{p,1}^{\frac{2}{p}}} dt \leq c, \quad i = 1, 2,$$

for some small enough constant c . Then the definition of A_i implies that

$$A_i(t, y) = \operatorname{Id} + \sum_{k=1}^{\infty} (-1)^k \left(\int_0^t Dv_i(\tau, y) d\tau \right)^k, \quad i = 1, 2.$$

Moreover, as proved in the appendix of [16], we have the following estimates:

$$\|A_i - \operatorname{Id}\|_{L_t^\infty(\dot{B}_{p,1}^{\frac{2}{p}})} \lesssim \|Dv_i\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}})}, \quad i = 1, 2, \quad (4.21)$$

$$\|\delta A\|_{L_t^\infty(\dot{B}_{p,1}^{\frac{2}{p}})} \lesssim \|D\delta v\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}})}, \quad (4.22)$$

$$\|\partial_t A_i\|_{\dot{B}_{p,1}^{\frac{2}{p}}} \lesssim \|Dv_i\|_{\dot{B}_{p,1}^{\frac{2}{p}}}, \quad i = 1, 2, \quad (4.23)$$

$$\|\partial_t \delta A\|_{L_t^2(\dot{B}_{p,1}^{\frac{2}{p}-1})} \lesssim \|v_1, v_2\|_{L_t^2(\dot{B}_{p,1}^{\frac{2}{p}})} \|D\delta v\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}})} + \|\delta v\|_{L_t^2(\dot{B}_{p,1}^{\frac{2}{p}})}. \quad (4.24)$$

We now choose m to be so large that

$$(1 + \|a_0\|_{\dot{B}_{p,1}^{\frac{2}{p}}})^3 \|b_0 - \dot{S}_m b_0, \lambda_0 - \dot{S}_m \lambda_0\|_{\dot{B}_{p,1}^{\frac{2}{p}}} \leq c_0,$$

where $b_0 \stackrel{\text{def}}{=} (1 + a_0)\tilde{\mu}(a_0) - \tilde{\mu}(0)$, $\lambda_0 \stackrel{\text{def}}{=} \int_0^{a_0} \tilde{\mu}(s)ds$. Then (3.28) is fulfilled so that we could apply Corollary 3.1 to (4.20) to obtain

$$\begin{aligned} & \|\delta v\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{2}{p}-1})} + \|\delta v\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}+1})} + \|\nabla \delta P\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} \\ & \lesssim \|\delta F\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} + \|R\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} + \|g\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}})} + 2^m \|\delta v\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}})}. \end{aligned} \quad (4.25)$$

On the other hand, it is easy to deduce from (4.20) that

$$\|\partial_t \delta v\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} \lesssim \|\delta v\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}+1})} + \|\nabla \delta P\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} + \|\delta F\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})}. \quad (4.26)$$

We denote $\delta E(t) \stackrel{\text{def}}{=} \|\delta v\|_{\tilde{L}_t^\infty(\dot{B}_{p,1}^{\frac{2}{p}-1})} + \|\Delta \delta v, \nabla \delta P, \partial_t \delta v\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})}$. Summing up (4.25) and (4.26) and using interpolation in Besov spaces leads to

$$\delta E(t) \lesssim \|\delta F\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} + \|R\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} + \|g\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}})} + 2^m t^{\frac{1}{2}} \delta E(t). \quad (4.27)$$

We shall prove that $\delta E(t) = 0$ for small enough t .

Now applying (4.21), (4.22) and product laws in Besov spaces, we arrive at

$$\begin{aligned} \|\delta F_1\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} & \lesssim \|v_2\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}+1})} \delta E(t), \\ \|\delta F_2\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} & \lesssim \|\nabla P_1\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} \delta E(t). \end{aligned}$$

Along the same line, one has

$$\|\delta F_3, \delta F_4, \delta F_5, \delta F_6\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} \lesssim \|v_1, v_2\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}+1})} \delta E(t).$$

Thus, we obtain

$$\|\delta F\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} \lesssim \|\Delta v_1, \Delta v_2, \nabla P_1\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} \delta E(t). \quad (4.28)$$

Using the same argument, we deduce from (4.21)–(4.24) that

$$\|R\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} \lesssim (\|\Delta v_2, \partial_t v_1\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} + \|v_1\|_{L_t^2(\dot{B}_{p,1}^{\frac{2}{p}})}) \delta E(t). \quad (4.29)$$

In order to bound δg in $L_t^1(\dot{B}_{p,1}^{\frac{2}{p}})$, we apply (4.19) to rewrite δg as follow:

$$g = \text{Tr}(D\delta v(\text{Id} - A_2) - Dv_1\delta A),$$

from which, we easily get that

$$\|g\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}})} \lesssim \|v_1, v_2\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}+1})} \delta E(t). \quad (4.30)$$

Plugging (4.28)–(4.30) into (4.27), we eventually get

$$\delta E(t) \lesssim \{\|\Delta v_1, \Delta v_2, \nabla P_1, \partial_t v_1\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} + \|v_1\|_{L_t^2(\dot{B}_{p,1}^{\frac{2}{p}})} + 2^m t^{\frac{1}{2}}\} \delta E(t),$$

from which, we get by taking t small enough that $\delta E(t) = 0$. The uniqueness on $[0, T]$ can be obtained by a standard argument.

5. Global well-posedness of (1.2) with homogeneous viscosity

In this section, we prove the global well-posedness part of Theorem 1.1.

5.1. Higher regularities of the solutions

Proposition 5.1. *Let $(a, u, \nabla \Pi)$ be the unique solution of (1.7) which satisfies (1.3), (4.17) and $1 + a \geq \kappa > 0$. Then for any $t_0 \in (0, T]$, there holds*

$$\begin{aligned} & \|u\|_{\tilde{L}^\infty([t_0, t]; \dot{B}_{p,1}^{\frac{2}{p}})} + \|u\|_{L^1([t_0, t]; \dot{B}_{p,1}^{\frac{2}{p}+2})} + \|\nabla \Pi\|_{L^1([t_0, t]; \dot{B}_{p,1}^{\frac{2}{p}})} \\ & \leq (t_0^{-\frac{1}{2}} + 2^m \|a\|_{L_t^\infty(\dot{B}_{p,1}^{\frac{2}{p}})}) e^{CZ(t)} \end{aligned} \quad (5.1)$$

with $Z(t) \stackrel{\text{def}}{=} \|u\|_{L_t^\infty(\dot{B}_{p,1}^{\frac{2}{p}-1})} + \|u\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}+1})} + \|\nabla \Pi\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})}$.

Proof. The proof of this proposition is similar to the proof of Proposition 3.2. For completeness, we outline its proof here. We first rewrite the momentum equation of (1.7) as

$$\partial_t u - (1 + \dot{S}_m a) \Delta u + (1 + \dot{S}_m a) \nabla \Pi = -u \cdot \nabla u + \dot{E}_m \quad (5.2)$$

with $\dot{E}_m \stackrel{\text{def}}{=} (a - \dot{S}_m a)(\Delta u - \nabla \Pi)$. Thanks to $\text{div } u = 0$, applying $\dot{\Delta}_j \mathcal{P}$ to (5.2) leads to

$$\begin{aligned} & \partial_t \dot{\Delta}_j u - \text{div}((1 + \dot{S}_m a) \dot{\Delta}_j \nabla u) \\ & = \dot{\Delta}_j \mathcal{P}(-u \cdot \nabla u + \dot{E}_m + \Pi \nabla \dot{S}_m a) - \dot{\Delta}_j(\nabla \dot{S}_m a \cdot \nabla u) \\ & \quad - \dot{\Delta}_j \mathcal{Q}(\dot{S}_m a \Delta u) + \text{div}([\dot{\Delta}_j, \dot{S}_m a] \nabla u), \end{aligned}$$

from which, we infer for $0 < \tau < t_0 \leq t \leq T$ that

$$\begin{aligned}
& \|u\|_{\tilde{L}^\infty([\tau,t];\dot{B}_{p,1}^{\frac{2}{p}})} + \|u\|_{L^1([\tau,t];\dot{B}_{p,1}^{\frac{2}{p}+2})} \\
& \lesssim \|u(\tau)\|_{\dot{B}_{p,1}^{\frac{2}{p}}} + \|u \cdot \nabla u\|_{L^1([\tau,t];\dot{B}_{p,1}^{\frac{2}{p}})} + \|\dot{E}_m\|_{L^1([\tau,t];\dot{B}_{p,1}^{\frac{2}{p}})} + \|\Pi \nabla \dot{S}_m a\|_{L^1([\tau,t];\dot{B}_{p,1}^{\frac{2}{p}})} \\
& \quad + \|\nabla \dot{S}_m a \cdot \nabla u\|_{L^1([\tau,t];\dot{B}_{p,1}^{\frac{2}{p}})} + \|\nabla \dot{S}_m a \cdot \Delta u\|_{L^1([\tau,t];\dot{B}_{p,1}^{\frac{2}{p}-1})} \\
& \quad + \sum_{j \in \mathbb{Z}} 2^{\frac{2j}{p}} \|\operatorname{div}([\dot{\Delta}_j, \dot{S}_m a] \nabla u)\|_{L^1([\tau,t];L^p)}, \tag{5.3}
\end{aligned}$$

where we used $\mathcal{Q}(\dot{S}_m a \Delta u) = -\nabla(-\Delta)^{-1}(\nabla \dot{S}_m a \cdot \Delta u)$. While applying $\dot{\Delta}_j \operatorname{div}$ to (5.2) gives

$$\operatorname{div}((1 + \dot{S}_m a) \dot{\Delta}_j \nabla \Pi) = \dot{\Delta}_j \operatorname{div}(-u \cdot \nabla u + \dot{E}_m + \dot{S}_m a \Delta u) - \operatorname{div}([\dot{\Delta}_j, \dot{S}_m a] \nabla \Pi),$$

which implies

$$\begin{aligned}
\|\nabla \Pi\|_{L^1([\tau,t];\dot{B}_{p,1}^{\frac{2}{p}})} & \lesssim \|u \cdot \nabla u\|_{L^1([\tau,t];\dot{B}_{p,1}^{\frac{2}{p}})} + \|\dot{E}_m\|_{L^1([\tau,t];\dot{B}_{p,1}^{\frac{2}{p}})} + \|\nabla \dot{S}_m a \cdot \Delta u\|_{L^1([\tau,t];\dot{B}_{p,1}^{\frac{2}{p}-1})} \\
& \quad + \sum_{j \in \mathbb{Z}} 2^{(\frac{2}{p}-1)j} \|\operatorname{div}([\dot{\Delta}_j, \dot{S}_m a] \nabla \Pi)\|_{L^1([\tau,t];L^p)}. \tag{5.4}
\end{aligned}$$

Summing up (5.3) and (5.4) and then applying product laws, commutator estimates (see [15, Lemma 5]) and (4.17), we arrive at

$$\begin{aligned}
& \|u\|_{\tilde{L}^\infty([\tau,t];\dot{B}_{p,1}^{\frac{2}{p}})} + \|u\|_{L^1([\tau,t];\dot{B}_{p,1}^{\frac{2}{p}+2})} + \|\nabla \Pi\|_{L^1([\tau,t];\dot{B}_{p,1}^{\frac{2}{p}})} \\
& \lesssim \|u(\tau)\|_{\dot{B}_{p,1}^{\frac{2}{p}}} + 2^m \|a\|_{L_t^\infty(\dot{B}_{p,1}^{\frac{2}{p}})} \|\Delta u, \nabla \Pi\|_{L_t^1(\dot{B}_{p,1}^{\frac{2}{p}-1})} + \int_\tau^t \|u\|_{\dot{B}_{p,1}^{\frac{2}{p}}} \|u\|_{\dot{B}_{p,1}^{\frac{2}{p}+1}} dt',
\end{aligned}$$

from which, we get by using Gronwall's inequality that

$$\begin{aligned}
& \|u\|_{\tilde{L}^\infty([\tau,t];\dot{B}_{p,1}^{\frac{2}{p}})} + \|u\|_{L^1([\tau,t];\dot{B}_{p,1}^{\frac{2}{p}+2})} + \|\nabla \Pi\|_{L^1([\tau,t];\dot{B}_{p,1}^{\frac{2}{p}})} \\
& \leq C(\|u(\tau)\|_{\dot{B}_{p,1}^{\frac{2}{p}}} + 2^m \|a\|_{L_t^\infty(\dot{B}_{p,1}^{\frac{2}{p}})} Z(t)) e^{CZ(t)}.
\end{aligned}$$

Integrating the above inequality for τ over $(0, t_0)$, we conclude the proof of (5.1). \square

5.2. Energy estimates in the L^2 framework

Let T^* be the maximal existence time of the unique local solution $(a, u, \nabla \Pi)$ obtained in Section 4. Thanks to (1.3) and (5.1), we infer that for any $t \in (0, T^*)$, there exists a time

$t_1 \in (0, t)$ such that $u(t_1) \in \dot{B}_{p,1}^{\frac{2}{p}-1}(\mathbb{R}^2) \cap \dot{B}_{p,1}^{\frac{2}{p}+2}(\mathbb{R}^2)$. As in (1.8), denote by $u_F(t) \stackrel{\text{def}}{=} e^{(t-t_1)\Delta} u(t_1)$.

Then thanks to Lemma 2.2, we have for $s \in [\frac{2}{p} - 1, \frac{2}{p} + 2]$ that

$$\|u_F\|_{\tilde{L}^\infty([t_1, \infty); \dot{B}_{p,1}^s)} + \|u_F\|_{L^1([t_1, \infty); \dot{B}_{p,1}^{s+2})} + \|u_F\|_{L^2([t_1, \infty); \dot{B}_{p,1}^{s+1})} \lesssim \|u(t_1)\|_{\dot{B}_{p,1}^s}. \quad (5.5)$$

Note that for $p \in (2, 4)$, the free solution u_F is not of finite energy. Fortunately, the convection term $u_F \cdot \nabla u_F$ is of finite energy. Indeed, for $u, v \in \dot{B}_{p,1}^{\frac{2}{p}-1}(\mathbb{R}^2) \cap \dot{B}_{p,1}^{\frac{2}{p}+1}(\mathbb{R}^2)$ with $p \in (2, 4)$, we have

$$\begin{aligned} \|u \cdot \nabla v\|_{L^2} &\leq \|u\|_{L^4} \|\nabla v\|_{L^4} \leq \|u\|_{\dot{B}_{p,1}^{\frac{3}{4}}}^{\frac{3}{4}} \|u\|_{\dot{B}_{p,1}^{\frac{2}{p}+1}}^{\frac{1}{4}} \|v\|_{\dot{B}_{p,1}^{\frac{3}{4}}}^{\frac{1}{4}} \|v\|_{\dot{B}_{p,1}^{\frac{2}{p}+1}}^{\frac{3}{4}} \\ &\leq \|u\|_{\dot{B}_{p,1}^{\frac{2}{p}-1}} \|v\|_{\dot{B}_{p,1}^{\frac{2}{p}+1}} + \|u\|_{\dot{B}_{p,1}^{\frac{2}{p}+1}} \|v\|_{\dot{B}_{p,1}^{\frac{2}{p}-1}}. \end{aligned} \quad (5.6)$$

Owing to (1.8), we next present the L^2 energy estimates for \bar{u} in the case when $p \in (2, 4)$. Similar estimates for $p \in (1, 2]$ will be mentioned after Lemma 5.3.

Lemma 5.1. (L^2 estimate of \bar{u}). *Under the assumptions of Theorem 1.1, there exists a time independent constant C such that*

$$\|\bar{u}\|_{L^\infty([t_1, T^*]; L^2)} + \|\nabla \bar{u}\|_{L^2([t_1, T^*]; L^2)} \leq C. \quad (5.7)$$

Proof. Firstly thanks $1 + a_0 \geq \kappa$, we deduce from the transport equation of (1.8) that

$$\frac{1}{1 + \|a_0\|_{L^\infty}} \leq \rho(t, x) \leq \frac{1}{\kappa}. \quad (5.8)$$

Next, taking the L^2 inner product of the \bar{u} equation of (1.8) with \bar{u} leads to

$$\frac{1}{2} \frac{d}{dt} \|\sqrt{\rho} \bar{u}\|_{L^2}^2 + \|\nabla \bar{u}\|_{L^2}^2 = \int_{\mathbb{R}^2} \bar{u} \cdot G dx \lesssim \|\sqrt{\rho} \bar{u}\|_{L^2} \|G\|_{L^2}. \quad (5.9)$$

Yet thanks to $1 - \rho = \rho a$ and (5.8), we get by applying Hölder inequality and (5.6) that

$$\begin{aligned} \|G\|_{L^2} &\lesssim \|a\|_{L^{\frac{2p}{p-2}}} \|\Delta u_F\|_{L^p} + \|u_F \cdot \nabla u_F\|_{L^2} + \|\sqrt{\rho} \bar{u}\|_{L^2} \|\nabla u_F\|_{L^\infty} \\ &\lesssim \|u_F\|_{\dot{B}_{p,1}^2} + \|u_F\|_{\dot{B}_{p,1}^{\frac{2}{p}-1}} \|u_F\|_{\dot{B}_{p,1}^{\frac{2}{p}+1}} + \|\sqrt{\rho} \bar{u}\|_{L^2} \|u_F\|_{\dot{B}_{p,1}^{\frac{2}{p}+1}}. \end{aligned} \quad (5.10)$$

Thanks to (5.5), plugging (5.10) into (5.9) and applying Gronwall's inequality gives rise to (5.7). This completes the proof of the lemma. \square

Lemma 5.2. (H^1 estimate of \bar{u}). Under the assumptions of Theorem 1.1, there exists a time independent constant C such that

$$\|\nabla \bar{u}\|_{L^\infty([t_1, T^*]; L^2)} + \|\bar{u}_t, \Delta \bar{u}, \nabla \Pi\|_{L^2([t_1, T^*]; L^2)} \leq C. \quad (5.11)$$

Proof. Taking the L^2 inner product of the \bar{u} equation of (1.8) with \bar{u}_t gives

$$\frac{d}{dt} \|\nabla \bar{u}\|_{L^2}^2 + \|\sqrt{\rho} \bar{u}_t\|_{L^2}^2 \lesssim \|G\|_{L^2}^2 + \|\rho(u_F + \bar{u}) \cdot \nabla \bar{u}\|_{L^2}^2. \quad (5.12)$$

On the other hand, we readily deduce from (1.8) that

$$\|\Delta \bar{u}\|_{L^2}^2 + \|\nabla \Pi\|_{L^2}^2 = \|\Delta \bar{u} - \nabla \Pi\|_{L^2}^2 \lesssim \|\sqrt{\rho} \bar{u}_t\|_{L^2}^2 + \|\rho(u_F + \bar{u}) \cdot \nabla \bar{u}\|_{L^2}^2 + \|G\|_{L^2}^2, \quad (5.13)$$

which along with (5.12) implies

$$\frac{d}{dt} \|\nabla \bar{u}\|_{L^2}^2 + C^{-1} \|\sqrt{\rho} \bar{u}_t, \Delta \bar{u}, \nabla \Pi\|_{L^2}^2 \lesssim \|G\|_{L^2}^2 + \|u_F\|_{L^\infty}^2 \|\nabla \bar{u}\|_{L^2}^2 + \|\bar{u} \cdot \nabla \bar{u}\|_{L^2}^2. \quad (5.14)$$

While taking advantage of Gagliardo-Nirenberg inequality, we obtain

$$\|\bar{u} \cdot \nabla \bar{u}\|_{L^2}^2 \lesssim \|\bar{u}\|_{L^4}^2 \|\nabla \bar{u}\|_{L^4}^2 \lesssim \|\bar{u}\|_{L^2} \|\nabla \bar{u}\|_{L^2}^2 \|\Delta \bar{u}\|_{L^2}. \quad (5.15)$$

Plugging (5.10) and (5.15) into (5.14) and using Young's inequality leads to

$$\begin{aligned} & \frac{d}{dt} \|\nabla \bar{u}\|_{L^2}^2 + C^{-1} \|\sqrt{\rho} \bar{u}_t, \Delta \bar{u}, \nabla \Pi\|_{L^2}^2 \\ & \lesssim (\|u_F\|_{\dot{B}_{p,1}^{\frac{2}{p}}}^2 + \|\bar{u}\|_{L^2}^2 \|\nabla \bar{u}\|_{L^2}^2) \|\nabla \bar{u}\|_{L^2}^2 + (\|u_F\|_{\dot{B}_{p,1}^{\frac{2}{p}-1}}^2 + \|\bar{u}\|_{L^2}^2) \|u_F\|_{\dot{B}_{p,1}^{\frac{2}{p}+1}}^2 + \|u_F\|_{\dot{B}_{p,1}^2}^2, \end{aligned}$$

from which, (5.5) and (5.7), we conclude the proof of (5.11). \square

Lemma 5.3. (H^2 estimate of \bar{u}). Under the assumptions of Theorem 1.1, there exists a time independent constant C such that for any $q \in [2, \infty)$

$$\|\bar{u}_t, \Delta \bar{u}, \nabla \Pi\|_{L^\infty([t_1, T^*]; L^2)} + \|\nabla \bar{u}_t\|_{L^2([t_1, T^*]; L^2)} + \|\Delta \bar{u}, \nabla \Pi\|_{L^2([t_1, T^*]; L^q)} \leq C. \quad (5.16)$$

Proof. Firstly, applying ∂_t to the \bar{u} equation of (1.8) and then taking the L^2 inner product of

the resulting equation with \bar{u}_t , we obtain

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|\sqrt{\rho} \bar{u}_t\|_{L^2}^2 + \|\nabla \bar{u}_t\|_{L^2}^2 \\
&= \int_{\mathbb{R}^2} (1 - \rho) \bar{u}_t \cdot \Delta^2 u_F dx \\
&\quad - \int_{\mathbb{R}^2} \rho_t \bar{u}_t \cdot (\bar{u}_t + \bar{u} \cdot \nabla \bar{u} + u_F \cdot \nabla \bar{u} + \bar{u} \cdot \nabla u_F + \Delta u_F + u_F \cdot \nabla u_F) dx \\
&\quad - \int_{\mathbb{R}^2} \rho \bar{u}_t \cdot (\bar{u}_t \cdot \nabla \bar{u} + \Delta u_F \cdot \nabla \bar{u} + \bar{u}_t \cdot \nabla u_F + \bar{u} \cdot \nabla \Delta u_F + \partial_t(u_F \cdot \nabla u_F)) dx \\
&\stackrel{\text{def}}{=} I_1 + I_2 + I_3.
\end{aligned} \tag{5.17}$$

Using $\rho_t = -\operatorname{div}(\rho(u_F + \bar{u}))$ and integration by parts gives

$$\begin{aligned}
I_2 &= -2 \int_{\mathbb{R}^2} \rho \bar{u}_t \cdot ((u_F + \bar{u}) \cdot \nabla \bar{u}_t) dx - \int_{\mathbb{R}^2} \rho (u_F + \bar{u})^j \bar{u}_t \cdot \{(u_F + \bar{u}) \cdot \nabla \partial_j \bar{u} \\
&\quad + \partial_j \bar{u} \cdot \nabla \bar{u} + \partial_j u_F \cdot \nabla \bar{u} + \partial_j \bar{u} \cdot \nabla u_F + \bar{u} \cdot \nabla \partial_j u_F + \partial_j \Delta u_F + \partial_j(u_F \cdot \nabla u_F)\} dx \\
&\quad - \int_{\mathbb{R}^2} \rho ((u_F + \bar{u}) \cdot \nabla \bar{u}_t) \cdot ((u_F + \bar{u}) \cdot \nabla \bar{u} + \bar{u} \cdot \nabla u_F + \Delta u_F + u_F \cdot \nabla u_F) dx \\
&\stackrel{\text{def}}{=} I_2^1 + I_2^2 + I_2^3.
\end{aligned}$$

Then it is easy to observe that

$$I_1 \lesssim \|a\|_{L^{\frac{2p}{p-2}}} \|\Delta^2 u_F\|_{L^p} \|\sqrt{\rho} \bar{u}_t\|_{L^2} \lesssim \|u_F\|_{\dot{B}_{p,1}^4} \|\sqrt{\rho} \bar{u}_t\|_{L^2}. \tag{5.18}$$

Applying Hölder's inequality and Gagliardo-Nirenberg inequality, we infer for any $\eta > 0$ that

$$\begin{aligned}
I_2^1 &\lesssim \|\sqrt{\rho} \bar{u}_t\|_{L^2} \|u_F\|_{L^\infty} \|\nabla \bar{u}_t\|_{L^2} + \|\bar{u}_t\|_{L^4} \|\bar{u}\|_{L^4} \|\nabla \bar{u}_t\|_{L^2} \\
&\lesssim \|\sqrt{\rho} \bar{u}_t\|_{L^2} \|u_F\|_{\dot{B}_{p,1}^{\frac{2}{p}}} \|\nabla \bar{u}_t\|_{L^2} + \|\sqrt{\rho} \bar{u}_t\|_{L^2}^{\frac{1}{2}} \|\bar{u}\|_{L^2}^{\frac{1}{2}} \|\nabla \bar{u}\|_{L^2}^{\frac{1}{2}} \|\nabla \bar{u}_t\|_{L^2}^{\frac{3}{2}} \\
&\lesssim \eta \|\nabla \bar{u}_t\|_{L^2}^2 + \frac{1}{\eta^3} (\|u_F\|_{\dot{B}_{p,1}^{\frac{2}{p}}}^2 + \|\bar{u}\|_{L^2}^2 \|\nabla \bar{u}\|_{L^2}^2) \|\sqrt{\rho} \bar{u}_t\|_{L^2}^2.
\end{aligned} \tag{5.19}$$

Yet thanks to (5.5), (5.7) and (5.11), we infer for any $p_1 \in [p, \infty]$ and $p_2 \in [2, \infty)$ that

$$\|u_F\|_{L^{p_1}} \leq C \|u_F\|_{\dot{B}_{p,1}^{\frac{2}{p} - \frac{2}{p_1}}} \leq C \quad \text{and} \quad \|\bar{u}\|_{L^{p_2}} \leq C \|\bar{u}\|_{H^1} \leq C,$$

which together with a similar argument as (5.19) results in

$$\begin{aligned}
I_2^2 &\lesssim \|\sqrt{\rho}\bar{u}_t\|_{L^2} \left\{ \|u_F + \bar{u}\|_{L^\infty}^2 \|\nabla^2 \bar{u}\|_{L^2} + \|u_F + \bar{u}\|_{L^6} \|\nabla \bar{u}\|_{L^6}^2 \right. \\
&\quad + \|u_F + \bar{u}\|_{L^4} \|\nabla u_F\|_{L^\infty} \|\nabla \bar{u}\|_{L^4} + \|u_F + \bar{u}\|_{L^4} \|\bar{u}\|_{L^4} \|\nabla^2 u_F\|_{L^\infty} \\
&\quad \left. + \|u_F + \bar{u}\|_{L^{\frac{2p}{p-2}}} \left(\|\nabla^3 u_F\|_{L^p} + \|\nabla u_F\|_{L^{2p}}^2 + \|u_F\|_{L^p} \|\nabla^2 u_F\|_{L^\infty} \right) \right\} \\
&\lesssim \|\sqrt{\rho}\bar{u}_t\|_{L^2} \left\{ \|u_F\|_{L^\infty} \|\Delta \bar{u}\|_{L^2} + \|\bar{u}\|_{L^2} \|\Delta \bar{u}\|_{L^2}^2 + \|\nabla \bar{u}\|_{L^2}^{\frac{2}{3}} \|\Delta \bar{u}\|_{L^2}^{\frac{4}{3}} \right. \\
&\quad + \|\nabla u_F\|_{L^\infty} \|\nabla \bar{u}\|_{L^2}^{\frac{1}{2}} \|\Delta \bar{u}\|_{L^2}^{\frac{1}{2}} + \|\nabla^2 u_F\|_{L^\infty} + \|\nabla^3 u_F\|_{L^p} + \|\nabla u_F\|_{L^{2p}}^2 \left. \right\} \\
&\lesssim \|\sqrt{\rho}\bar{u}_t\|_{L^2} \left\{ \|\Delta \bar{u}\|_{L^2}^2 + \|\nabla \bar{u}\|_{L^2}^2 + \|u_F\|_{\dot{B}_{p,1}^{\frac{2}{p}}}^2 + \|u_F\|_{\dot{B}_{p,1}^{\frac{2}{p}+1}}^2 + \|u_F\|_{\dot{B}_{p,1}^{\frac{2}{p}+2}} \right. \\
&\quad \left. + \|u_F\|_{\dot{B}_{p,1}^3} + \|u_F\|_{\dot{B}_{p,1}^{\frac{1}{p}+1}}^2 \right\}. \tag{5.20}
\end{aligned}$$

Along the same line, one has

$$\begin{aligned}
I_2^3 &\lesssim \|\nabla \bar{u}_t\|_{L^2} \left\{ \|u_F + \bar{u}\|_{L^6}^2 \|\nabla \bar{u}\|_{L^6} + \|u_F + \bar{u}\|_{L^4} \|\bar{u}\|_{L^4} \|\nabla u_F\|_{L^\infty} \right. \\
&\quad \left. + \|u_F + \bar{u}\|_{L^{\frac{2p}{p-2}}} \left(\|\Delta u_F\|_{L^p} + \|u_F\|_{L^p} \|\nabla u_F\|_{L^\infty} \right) \right\} \\
&\lesssim \eta \|\nabla \bar{u}_t\|_{L^2}^2 + \frac{1}{\eta} \left(\|\nabla \bar{u}\|_{L^2}^2 + \|\Delta \bar{u}\|_{L^2}^2 + \|u_F\|_{\dot{B}_{p,1}^{\frac{2}{p}+1}}^2 + \|u_F\|_{\dot{B}_{p,1}^2}^2 \right). \tag{5.21}
\end{aligned}$$

Finally thanks to (5.6), (5.7) and (5.11), using the same argument leads to

$$\begin{aligned}
I_3 &\lesssim \|\bar{u}_t\|_{L^4}^2 \|\nabla \bar{u}\|_{L^2} + \|\sqrt{\rho}\bar{u}_t\|_{L^2} \|\Delta u_F\|_{L^\infty} \|\nabla \bar{u}\|_{L^2} + \|\sqrt{\rho}\bar{u}_t\|_{L^2}^2 \|\nabla u_F\|_{L^\infty} \\
&\quad + \|\sqrt{\rho}\bar{u}_t\|_{L^2} \|\bar{u}\|_{L^2} \|\nabla \Delta u_F\|_{L^\infty} + \|\sqrt{\rho}\bar{u}_t\|_{L^2} \|\partial_t(u_F \cdot \nabla u_F)\|_{L^2} \\
&\lesssim \|\bar{u}_t\|_{L^2} \|\nabla \bar{u}_t\|_{L^2} \|\nabla \bar{u}\|_{L^2} + \|\sqrt{\rho}\bar{u}_t\|_{L^2}^2 \|\nabla u_F\|_{L^\infty} \\
&\quad + \|\sqrt{\rho}\bar{u}_t\|_{L^2} \left(\|\Delta u_F\|_{L^\infty} + \|\nabla \Delta u_F\|_{L^\infty} + \|u_F\|_{\dot{B}_{p,1}^{\frac{2}{p}+3}} \|u_F\|_{\dot{B}_{p,1}^{\frac{2}{p}-1}} \right) \\
&\lesssim \eta \|\nabla \bar{u}_t\|_{L^2}^2 + \|\sqrt{\rho}\bar{u}_t\|_{L^2}^2 \left(\frac{1}{\eta} \|\nabla \bar{u}\|_{L^2}^2 + \|u_F\|_{\dot{B}_{p,1}^{\frac{2}{p}+1}} \right) \\
&\quad + \|\sqrt{\rho}\bar{u}_t\|_{L^2} \left(\|u_F\|_{\dot{B}_{p,1}^{\frac{2}{p}+2}} + \|u_F\|_{\dot{B}_{p,1}^{\frac{2}{p}+3}} \right). \tag{5.22}
\end{aligned}$$

Thus, plugging (5.18)–(5.22) into (5.17) and taking η small enough yields

$$\begin{aligned}
&\frac{d}{dt} \|\sqrt{\rho}\bar{u}_t\|_{L^2}^2 + \|\nabla \bar{u}_t\|_{L^2}^2 \\
&\lesssim \|\sqrt{\rho}\bar{u}_t\|_{L^2} \left(\|\Delta \bar{u}\|_{L^2}^2 + \|\nabla \bar{u}\|_{L^2}^2 + \|u_F\|_{\dot{B}_{p,1}^{\frac{2}{p}+1}}^2 + \|u_F\|_{\dot{B}_{p,1}^4} \right) \\
&\quad + \|\sqrt{\rho}\bar{u}_t\|_{L^2}^2 \left(\|\nabla \bar{u}\|_{L^2}^2 + \|u_F\|_{\dot{B}_{p,1}^{\frac{2}{p}+1}}^2 \right) + \|\nabla \bar{u}\|_{L^2}^2 + \|\Delta \bar{u}\|_{L^2}^2 + \|u_F\|_{\dot{B}_{p,1}^{\frac{2}{p}+1}}^2 + \|u_F\|_{\dot{B}_{p,1}^2}^2 \\
&\lesssim (1 + \|\sqrt{\rho}\bar{u}_t\|_{L^2}^2) \left(\|\Delta \bar{u}\|_{L^2}^2 + \|\nabla \bar{u}\|_{L^2}^2 + \|u_F\|_{\dot{B}_{p,1}^{\frac{2}{p}+1}}^2 + \|u_F\|_{\dot{B}_{p,1}^4} \right). \tag{5.23}
\end{aligned}$$

where we used the fact that $\|u_F\|_{\dot{B}_{p,1}^s} \leq \|u_F\|_{\dot{B}_{p,1}^{s_1}} + \|u_F\|_{\dot{B}_{p,1}^{s_2}}$ for $s \in [s_1, s_2]$.

Whereas taking the L^2 inner product of the \bar{u} equation of (1.8) with \bar{u}_t at $t = t_1$ and using (5.5) and (5.6) results in

$$\begin{aligned} \|(\sqrt{\rho}\bar{u}_t)(t_1)\|_{L^2} &\leq C\|a(t_1)\|_{L^{\frac{2p}{p-2}}} \|\Delta u_F(t_1)\|_{L^p} + C\|u_F \cdot \nabla u_F(t_1)\|_{L^2} \\ &\leq C\|u(t_1)\|_{\dot{B}_{p,1}^2} + C\|u(t_1)\|_{\dot{B}_{p,1}^{\frac{2}{p}-1}} \|u(t_1)\|_{\dot{B}_{p,1}^{\frac{2}{p}+1}} \leq C. \end{aligned}$$

As a consequence, applying Gronwall's inequality to (5.23) and taking advantage of (5.5), (5.7) and (5.11) gives rise to

$$\|\bar{u}_t\|_{L^\infty([t_1, T^*]; L^2)} + \|\nabla \bar{u}_t\|_{L^2([t_1, T^*]; L^2)} \leq C. \quad (5.24)$$

Owing to (1.8), we can derive the second space derivative estimate of \bar{u} . Indeed, we deduce from (5.10), (5.13) and (5.15) that

$$\begin{aligned} \|\Delta \bar{u}\|_{L^2}^2 + \|\nabla \Pi\|_{L^2}^2 &\lesssim \|\bar{u}_t\|_{L^2}^2 + (1 + \|\nabla \bar{u}\|_{L^2}^2) \|u_F\|_{\dot{B}_{p,1}^{\frac{2}{p}-1}} \|u_F\|_{\dot{B}_{p,1}^{\frac{2}{p}+1}} \\ &\quad + \|\bar{u}\|_{L^2}^2 \|\nabla \bar{u}\|_{L^2}^4 + \|u_F\|_{\dot{B}_{p,1}^2}^2 + \|\bar{u}\|_{L^2} \|\nabla \bar{u}\|_{L^2}^{\frac{2}{p}+1}, \end{aligned}$$

which along with (5.5), (5.7), (5.11) and (5.24) ensures that

$$\|\Delta \bar{u}\|_{L^\infty([t_1, T^*]; L^2)} + \|\nabla \Pi\|_{L^\infty([t_1, T^*]; L^2)} \leq C. \quad (5.25)$$

While thanks to $\operatorname{div} \bar{u} = 0$, we deduce from the \bar{u} equation of (1.8) for $q \in [p, \infty)$ that

$$\begin{aligned} \|\Delta \bar{u}\|_{L^q} + \|\nabla \Pi\|_{L^q} &\lesssim \|\bar{u}_t\|_{L^q} + \|u_F + \bar{u}\|_{L^{2q}} \|\nabla(u_F + \bar{u})\|_{L^{2q}} + \|\Delta u_F\|_{L^q} \\ &\lesssim \|\bar{u}_t\|_{L^2}^{\frac{2}{q}} \|\nabla \bar{u}_t\|_{L^2}^{1-\frac{2}{q}} + \|\nabla \bar{u}\|_{L^2}^{\frac{1}{q}} \|\Delta \bar{u}\|_{L^2}^{1-\frac{1}{q}} \\ &\quad + \|u_F\|_{\dot{B}_{p,1}^{1+\frac{2}{p}-\frac{1}{q}}} + \|u_F\|_{\dot{B}_{p,1}^{2(1+\frac{1}{p}-\frac{1}{q})}}, \end{aligned}$$

which along with (5.5), (5.7), (5.11) and (5.24) implies

$$\|\Delta \bar{u}\|_{L^2([t_1, T^*]; L^q)} + \|\nabla \Pi\|_{L^2([t_1, T^*]; L^q)} \leq C. \quad (5.26)$$

Combining (5.24)–(5.26) and (5.11) gives the results. \square

Remark 5.1. For $p \in (1, 2]$, we have $\dot{B}_{p,1}^{\frac{2}{p}-1}(\mathbb{R}^2) \hookrightarrow L^2(\mathbb{R}^2)$. Instead of using the decomposition $u = u_F + \bar{u}$, we could directly present the L^2 energy estimates for u . With some slight modifications of the proof of Lemmas 5.1–5.3, we could deduce from (1.6) for $q \in [2, \infty)$ that

$$\|u_t, \Delta u, \nabla \Pi\|_{L^\infty([t_1, T^*]; L^2)} + \|\nabla u_t\|_{L^2([t_1, T^*]; L^2)} + \|\Delta u, \nabla \Pi\|_{L^2([t_1, T^*]; L^q)} \leq C. \quad (5.27)$$

5.3. Proof of the global well-posedness part of Theorem 1.1

Firstly, thanks to Lemma 2.1, one has for any $q \in (2, \infty)$

$$\|\bar{u}\|_{\dot{B}_{q,1}^{\frac{2}{q}+1}} \lesssim \|\nabla \bar{u}\|_{L^q}^{1-\frac{2}{q}} \|\Delta \bar{u}\|_{L^q}^{\frac{2}{q}} \lesssim \|\nabla \bar{u}\|_{H^1}^{1-\frac{2}{q}} \|\Delta \bar{u}\|_{L^q}^{\frac{2}{q}},$$

which together with (5.5), (5.7), (5.11), (5.16) and (5.27) results in

$$\|u\|_{L^1([t_1, t]; \dot{B}_{q,1}^{\frac{2}{q}+1})} \leq C t^{\frac{1}{2}}, \quad t < T^*.$$

Then for $q \in (2, \infty)$ with $\frac{1}{q} \geq \frac{1}{p} - \frac{1}{2}$, we get by applying Proposition 2.1 to the transport equation of (1.7) that

$$\|a\|_{\tilde{L}^\infty([t_1, t]; \dot{B}_{p,1}^{\frac{2}{p}})} \leq \|a(t_1)\|_{\dot{B}_{p,1}^{\frac{2}{p}}} \exp \left(C \|u\|_{L^1([t_1, t]; \dot{B}_{q,1}^{\frac{2}{q}+1})} \right) \leq C \exp(C t^{\frac{1}{2}}). \quad (5.28)$$

On the other hand, we rewrite the equation for u in (1.7) as

$$\partial_t u - \Delta u + \nabla \Pi = \frac{a}{1+a} \partial_t u - \frac{1}{1+a} u \cdot \nabla u.$$

Thanks to $\operatorname{div} u = 0$, it is easy to observe that for $t \in [t_1, T^*)$

$$\begin{aligned} & \|u\|_{\tilde{L}^\infty([t_1, t]; \dot{B}_{p,1}^{\frac{2}{p}-1})} + \|u\|_{L^1([t_1, t]; \dot{B}_{p,1}^{\frac{2}{p}+1})} + \|\nabla \Pi\|_{L^1([t_1, t]; \dot{B}_{p,1}^{\frac{2}{p}-1})} \\ & \lesssim \|u(t_1)\|_{\dot{B}_{p,1}^{\frac{2}{p}-1}} + \left\| \frac{a}{1+a} \partial_t u \right\|_{L^1([t_1, t]; \dot{B}_{p,1}^{\frac{2}{p}-1})} + \left\| \frac{1}{1+a} u \cdot \nabla u \right\|_{L^1([t_1, t]; \dot{B}_{p,1}^{\frac{2}{p}-1})}. \end{aligned} \quad (5.29)$$

Yet applying Lemma 2.3 leads to

$$\begin{aligned} \left\| \frac{1}{1+a} u \cdot \nabla u \right\|_{L^1([t_1, t]; \dot{B}_{p,1}^{\frac{2}{p}-1})} & \leq C (1 + \|a\|_{L^\infty([t_1, t]; \dot{B}_{p,1}^{\frac{2}{p}})}) \int_{t_1}^t \|u\|_{\dot{B}_{p,1}^{\frac{2}{p}-1}} \|u\|_{\dot{B}_{q,1}^{\frac{2}{q}+1}} dt' \\ & \leq C \exp(C t^{\frac{1}{2}}) \int_{t_1}^t \|u\|_{\dot{B}_{p,1}^{\frac{2}{p}-1}} \|u\|_{\dot{B}_{q,1}^{\frac{2}{q}+1}} dt'. \end{aligned} \quad (5.30)$$

where $q \in (2, \infty)$ satisfies $\frac{1}{q} > \frac{1}{2} - \frac{1}{p}$ and $\frac{1}{q} \geq \frac{1}{p} - \frac{1}{2}$. To control the second term in the right hand side of (5.29), we use again Lemma 2.3 to get for $p \in (1, 2]$ and $\alpha \in (\frac{2}{p} - 1, 1)$ that

$$\begin{aligned} \left\| \frac{a}{1+a} \partial_t u \right\|_{L^1([t_1, t]; \dot{B}_{p,1}^{\frac{2}{p}-1})} & \leq C \|a\|_{L^\infty([t_1, t]; \dot{B}_{p,1}^{\frac{2}{p}-\alpha})} \|\partial_t u\|_{L^1([t_1, t]; \dot{B}_{2,1}^\alpha)} \\ & \leq C t^{\frac{1}{2}} \|a\|_{L^\infty([t_1, t]; \dot{B}_{p,1}^{\frac{2}{p}})} \|\partial_t u\|_{L^2([t_1, t]; H^1)} \leq C \exp(C t^{\frac{1}{2}}). \end{aligned} \quad (5.31)$$

While for $p \in (2, 4)$ and $\alpha \in (0, \frac{2}{p})$, we alternatively get

$$\begin{aligned}
& \left\| \frac{a}{1+a} \partial_t u \right\|_{L^1([t_1, t]; \dot{B}_{p,1}^{\frac{2}{p}-1})} \\
& \leq C \|a\|_{L^\infty([t_1, t]; \dot{B}_{p,1}^{\frac{2}{p}})} \|u_F\|_{L^1([t_1, t]; \dot{B}_{p,1}^{\frac{2}{p}+1})} + C \|a\|_{L^\infty([t_1, t]; \dot{B}_{p,1}^{\frac{2}{p}-\alpha})} \|\partial_t \bar{u}\|_{L^1([t_1, t]; \dot{B}_{2,1}^\alpha)} \\
& \leq C \exp(Ct^{\frac{1}{2}}).
\end{aligned} \tag{5.32}$$

Plugging (5.30)–(5.32) into (5.29) and then applying Gronwall's inequality leads to

$$\|u\|_{\tilde{L}^\infty([t_1, t]; \dot{B}_{p,1}^{\frac{2}{p}-1})} + \|u\|_{L^1([t_1, t]; \dot{B}_{p,1}^{\frac{2}{p}+1})} + \|\nabla \Pi\|_{L^1([t_1, t]; \dot{B}_{p,1}^{\frac{2}{p}-1})} \leq C \exp \left\{ C \exp(Ct^{\frac{1}{2}}) \right\}. \tag{5.33}$$

From (5.28) and (5.33), we infer by a standard argument that $T^* = \infty$.

The proof of Theorem 1.1 is completed.

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